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Ph.D. Thesis

Properties of minimally tough graphs

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Chapter 1

Introduction

All graphs considered in this work are finite, simple and undirected.

The notion of toughness was introduced by Chvátal [13] to investigate Hamiltonicity. Hamilton cycles have always been well-studied, in particular, one of Karp's 21 NP-complete problems is to decide whether a graph contains a Hamiltonian cycle [19].

There are many either necessary or sufficient conditions for a graph to be Hamiltonian. The commonly known sufficient conditions concern the degree sequence of the graph. For example Dirac's theorem [14] says that every graph on $n \geq 3$ vertices with minimum degree at least $n/2$ contains a Hamiltonian cycle. A quite evident necessary condition gave the idea of defining graph toughness: if a graph contains a Hamiltonian cycle, then the removal of some vertices can leave at most as many components as the number of the removed vertices. Graphs with this latter property are called 1-tough, and in general, a graph is called t-tough (where t is a positive real number) if the removal of any vertex set S leaves at most $|S|/t$ components provided the removal of S disconnects the graphs, and all graphs are considered 0-tough. The toughness of a graph is the largest t for which the graph is t-tough, whereby the toughness of complete graphs is defined as infinity. For instance, the toughness of nonconnected graphs is 0, the toughness of cycles of length at least four is 1, but the cycle of length three is a complete graph, thus its toughness is infinity by definition.

While it is easy to see that not every 1-tough graph contains a Hamiltonian cycle (a well-known counterexample is the Petersen graph), Chvátal conjectured in his first article about graph toughness [13] that there exists a positive real number t_0 such that every t_0-tough graph is Hamiltonian. His stronger conjecture was that every more than 3/2-tough graph is Hamiltonian – but this was disproved by Thomassen [12]. Thereafter it was conjectured (based on [15]) that every 2-tough graph is Hamiltonian – but this was also disproved, this time by Bauer et al. [7]. Actually, they showed that for any $\varepsilon > 0$ there exists a $(9/4 - \varepsilon)$-tough graph that does not even contain a Hamiltonian path, implying if Chvátal's t_0-conjecture is true, then $t_0 \geq 9/4$ holds. The conjecture is still open, but there are some positive partial results in some graph classes: every 1-tough interval graph is Hamiltonian [20], so is every 3/2-tough split graph [21] and every 10-tough chordal graph [17], to name but a few.

This thesis mostly focuses on minimally tough graphs. It is easy to see that the more edges a graph has, the larger its toughness can be. A graph is called minimally t-tough (where t is a positive real number or inftinity) if the toughness of the graph is exactly t but the removal of any edge decreases the toughness. For instance, every complete graph on at least two vertices is minimally ∞-tough and every cycle of length at least four is minimally 1-tough.

It follows directly from the definition of toughness that every t-tough non-complete graph is $2t$-connected, thus the minimum degree of any t-tough non-complete graph is at least $\lceil 2t \rceil$ (where t is a nonnegative real number). Motivated by a theorem of Mader [24] stating that every minimally k-connected graph has a vertex of degree k (where k is a positive integer), there is a conjecture regarding the minimum degree of minimally tough graphs. This conjecture appeared in writing only for $t = 1$ under the name of Kriesell [18], but can be naturally generalized for any positive real number t: every minimally t-tough graph has a vertex of degree $\lceil 2t \rceil$. Since a minimally tough graph is not necessarily minimally connected (see Figure 3.1 for an example), Kriesell's conjecture does not follow from Mader's theorem directly.

Since every Hamiltonian graph is 1-tough (and the toughness of K_3 is infinity), the only minimally 1-tough Hamiltonian graphs are cycles of length at least 4. Thus, the above mentioned theorem of Dirac [14] provides a trivial upper bound on the minimum degree of minimally 1-tough graphs on n vertices: it is less than $n/2$, except for the cycle of length 4. After introducing the necessary definitions and collecting some preliminary results in Chapter 2, we present an improvement on this upper bound by a constant factor in Chapter 3 (based on [3]): we prove that every minimally 1-tough graph on n vertices has a vertex of degree at most $n/3 + 1$.

In [8], Bauer et al. proved that recognizing t-tough graphs is coNP-hard, and in Chapter 4 (based on [2]) we show that recognizing minimally t-tough graphs is DP-hard. The complexity class DP was introduced by Papadimitriou and Yannakakis in [26] since extremal problems usually seem not to belong to NP $\cup$ coNP. A language L belongs to the class DP if it can be expressed as the intersection of a language in NP and another one in coNP.

Finally, in Chapter 5 (based on [4]) we study bipartite graphs. Although the toughness of any bipartite graph, except for the graphs K_1 and K_2, is at most one, recognizing 1-tough bipartite graphs does not become easier than recognizing 1-tough graphs in general: Kratsch et al. proved that this problem is still coNP-hard [21]. In this chapter, we extend this theorem to any positive rational number $t \leq 1$. Moreover, we also prove that for any fixed integer $k \geq 2$ and positive rational number $t \leq 1$, recognizing t-tough k-connected bipartite graphs is also coNP-hard and so is recognizing 1-tough at least 6-regular bipartite graphs. Furthermore, we also give a stronger upper bound on the minimum degree of minimally 1-tough, bipartite graphs, than the one we gave earlier in general: we prove that every minimally 1-tough, bipartite graph on n vertices has a vertex of degree at most $(n + 6)/4$.

Chapter 2

Preliminaries

In this chapter, we present some necessary definitions and claims.

Let $\omega(G)$ denote the *number of components*, $\alpha(G)$ the *independence number*, $\kappa(G)$ the *connectivity number* and $\delta(G)$ the *minimum degree* of a graph G. For a vertex v of a graph G, the *degree* of v is denoted by $d(v)$. For a graph G and a vertex set $V' \subseteq V(G)$, let $G[V']$ denote the *subgraph of G induced by V'*. For a connected graph G, a vertex set $S \subseteq V(G)$ is called a *cutset* if its removal disconnects the graph.

(Using $\omega(G)$ to denote the number of components might be confusing; most of the literature on toughness, however, uses this notation.)

2.1 Toughness

The notion of toughness was introduced by Chvátal [13] to investigate Hamiltonicity.

Definition 2.1. Let t be a real number. A graph G is called *t-tough* if $|S| \geq t\omega(G - S)$ holds for any vertex set $S \subseteq V(G)$ that disconnects the graph (i.e. for any $S \subseteq V(G)$ with $\omega(G - S) > 1$). The *toughness* of G, denoted by $\tau(G)$, is the largest t for which G is t-tough, taking $\tau(K_n) = \infty$ for all $n \geq 1$.

We say that a cutset $S \subseteq V(G)$ is a *tough set* if $\omega(G - S) = |S|/\tau(G)$.

The following proposition is a simple observation.

Proposition 2.2. Let $t \leq 1$ be a positive rational number and G a t-tough graph. Then

$$\omega(G - S) \leq |S|/t$$

for any nonempty proper subset S of $V(G)$.

Proof. If S is a cutset in G, then by the definition of toughness

$$\omega(G - S) \leq |S|/t$$

holds.

If S is not a cutset in G, then $\omega(G - S) = 1$ since $S \neq V(G)$. On the other hand, $|S|/t \geq 1$ since $S \neq \emptyset$ and $t \leq 1$. Therefore,

$$\omega(G - S) \leq |S|/t$$

holds in this case as well. □

As is clear from its proof, the above proposition holds even if S is not a cutset. However, it does not necessarily hold if $t > 1$ and S is not a cutset: if $t > 1$, then the graph cannot contain a cut-vertex; therefore $\omega(G - S) = 1$ for any subset S with $|S| = 1$, while $|S|/t = 1/t < 1$.

2.2 Minimally toughness

Obviously, the more edges a graph has, the larger its toughness can be. The main focus of this work is on the graphs whose toughness decreases whenever any of their edges is deleted.

Definition 2.3. A graph G is said to be *minimally t-tough* if $\tau(G) = t$ and $\tau(G - e) < t$ for all $e \in E(G)$.

The following proposition describes the basic structure of minimally tough graphs.

Proposition 2.4. Let t be a positive rational number and G a minimally t-tough graph. For every edge e of G,

- the edge e is a bridge in G, or

– there exists a vertex set $S = S(e) \subseteq V(G)$ with

$$\omega(G - S) \leq \frac{|S|}{t} \quad \text{and} \quad \omega\big((G - e) - S\big) > \frac{|S|}{t},$$

and the edge e is a bridge in $G - S$.

In the first case, we define $S = S(e) = \emptyset$.

Proof. Let e be an arbitrary edge of G which is not a bridge. Since G is minimally t-tough, $\tau(G - e) < t$. Since e is not a bridge, $G - e$ is still connected, so there exists a cutset $S = S(e) \subseteq V(G - e) = V(G)$ in $G - e$ satisfying

$$\omega\big((G - e) - S\big) > |S|/t.$$

By Proposition 2.2, if $t \leq 1$, then $\omega(G - S) \leq |S|/t$. So assume that $t > 1$. Now there are two cases.

Case 1: ($t > 1$ and) S is a cutset in G.
Since $\tau(G) = t$ and S is a cutset, $\omega(G - S) \leq |S|/t$. This is only possible if e connects two components of $(G - e) - S$, i.e., if e is a bridge in $G - S$.

Case 2: ($t > 1$ and) S is not a cutset in G.
Then $\omega(G - S) = 1$. Since S is a cutset in $G - e$, the edge e must connect two components of $(G - e) - S$, so e is a bridge in $G - S$ and

$$\omega\big((G - e) - S\big) = 2.$$

Now we show that $\omega(G - S) \leq |S|/t$. Suppose to the contrary that $\omega(G - S) > |S|/t$. Since $\omega(G - S) = 1$, this implies that $|S| < t$. Moreover, since $\tau(G) = t$, the graph G is $\lceil 2t \rceil$-connected, thus it has at least $2t + 1$ vertices. From this it follows that S and one of the endpoints of e form a cutset in G, otherwise G would only have

$$|S| + 2 < t + 2 < 2t + 1$$

vertices (where the latter inequality is valid since $t > 1$). Let S' denote this cutset. Since G is t-tough and S' is a cutset in G,

$$2 \leq \omega(G - S') \leq \frac{|S'|}{t} = \frac{|S| + 1}{t},$$

so $|S| \geq 2t - 1$. Therefore

$$2t - 1 \leq |S| < t,$$

which implies that $t < 1$ and that is a contradiction.

So in either case

$$\omega(G-S) \leq \frac{|S|}{t} \quad \text{and} \quad \omega\big((G-e)-S\big) > \frac{|S|}{t},$$

and e is a bridge in $G-S$.

$\square$

2.3 Almost minimally 1-tough graphs

The graphs K_2 and K_3 behave similarly as minimally 1-tough graphs: they are 1-tough, and the removal of any of their edges decreases their toughness below 1. However, they are not minimally 1-tough since their toughness is infinity. To handle these kinds of graphs, we introduce the following definition.

Definition 2.5. A graph G is *almost minimally* 1*-tough* if $\tau(G) \geq 1$ and $\tau(G-e) < 1$ for all $e \in E(G)$.

In fact, the only almost minimally 1-tough graphs are minimally 1-tough graphs and the graphs K_2 and K_3.

Claim 2.6. For a graph G the following are equivalent.

(1) The graph G is almost minimally 1-tough.

(2) The graph G is 1-tough and for every $e \in E(G)$, the edge e is a bridge or there exists a vertex set $S = S(e) \subseteq V(G)$ with

$$\omega(G-S) = |S| \qquad \text{and} \qquad \omega\big((G-e)-S\big) = |S|+1.$$

(If e is a bridge, we define $S = S(e) = \emptyset$.)

(3) The graph G is either minimally 1-tough or $G \simeq K_2$ or $G \simeq K_3$.

Proof.

$(1) \implies (2)$: Let e be an arbitrary edge of G, and let us assume that it is not a bridge. Since $\tau(G-e) < 1$ and $G-e$ is still connected, there exists a cutset $S = S(e) \subseteq V(G-e) = V(G)$ in $G-e$ satisfying $\omega\big((G-e)-S\big) > |S|$.

Now there are two cases.

Case 1: S is a cutset in G.

Since $\tau(G) \geq 1$ and S is a cutset, $\omega(G - S) \leq |S|$. This is only possible if e connects two components of $(G - e) - S$, which means that

$$\omega\big((G - e) - S\big) = |S| + 1 \qquad \text{and} \qquad \omega(G - S) = |S|.$$

Case 2: S is not a cutset in G.

Then $\omega(G - S) = 1$. On the other hand,

$$\omega\big((G - e) - S\big) \geq 2$$

since S is a cutset in $G - e$. This is only possible if e connects two components of $(G - e) - S$, which means that

$$\omega\big((G - e) - S\big) = 2.$$

Since

$$\omega\big((G - e) - S\big) > |S|,$$

this implies that $|S| \leq 1$. Moreover, $|S| = 1$ since e is not a bridge in G. Hence,

$$\omega\big((G - e) - S\big) = |S| + 1 \qquad \text{and} \qquad \omega(G - S) = |S|.$$

$(2) \implies (3)$: Then $\tau(G) \geq 1$ and $\tau(G - e) < 1$ for every $e \in E(G)$. Let us assume that G is not minimally 1-tough, i.e. $\tau(G) > 1$. We need to show that $G \simeq K_2$ or $G \simeq K_3$.

Suppose to the contrary that G has at least 4 vertices. Let $e \in E(G)$ be an arbitrary edge, and let $S = S(e) \subseteq V(G)$ be a vertex set for which

$$\omega(G - S) = S \qquad \text{and} \qquad \omega\big((G - e) - S\big) = |S| + 1.$$

Since $\tau(G) > 1$ and $\omega(G - S) = |S|$, the vertex set S cannot be a cutset in G, so $|S| \leq 1$ must hold. Since G has at least 4 vertices, S and one of the endpoints of e form a cutset of size at most 2, so $\tau(G) \leq 1$, which is a contradiction. This means that $G \simeq K_2$ or $G \simeq K_3$ since there are no other 1-tough graphs on at most 3 vertices with at least one edge.

$(3) \implies (1)$: Trivial. $\square$

Proposition 2.7. Let G be an almost minimally 1-tough graph. Then

$$\omega(G - S) \leq |S|$$

for any proper subset S of $V(G)$.

Proof. By Claim 2.6, the graph G is either minimally 1-tough or $G \simeq K_2$ or $G \simeq K_3$. If G is minimally 1-tough, then $\tau(G) = 1$, and we already covered this case in Proposition 2.2. If $G \simeq K_2$ or $G \simeq K_3$, then $\omega(G - S) = 1$ and $|S| = 1$ hold for any proper subset S of $V(G)$. □

2.4 Complexity

The complexity of recognizing t-tough graphs has also been in the interest of research.

Let t be an arbitrary positive rational number and consider the following problem.

***t*-TOUGH**
Instance: a graph G.
Question: is it true that $\tau(G) \geq t$?

Note that in this problem, t is not part of the input.

It is easy to see that for any positive real number t, the problem t-TOUGH is in coNP: if a graph G is not t-tough, then a witness is a vertex set $S \subseteq V(G)$ whose removal disconnects the graph and leaves more than $|S|/t$ components. By reducing a variant of the independent set problem to the complement of t-TOUGH, Bauer et al. proved the following.

Theorem 2.8 (Bauer, Hakimi, Schmeichel, [8])**.** The problem t-TOUGH is coNP-complete for any positive rational number t.

Bauer et al. also proved the following.

Theorem 2.9 (Bauer, van den Heuvel, Morgana, Schmeichel, [10])**.** For any fixed integer $r \geq 3$, the problem 1-TOUGH is coNP-complete for r-regular graphs.

Although the toughness of any bipartite graph, except for the graphs K_1 and K_2, is at most one, the problem 1-TOUGH does not become easier for bipartite graphs.

Theorem 2.10 (Kratsch, Lehel, Müller, [21])**.** The problem 1-TOUGH is coNP-complete for bipartite graphs.

However, in some graph classes the toughness can be computed in polynomial time, for instance, in the class of split graphs.

Theorem 2.11 (Woeginger, [27])**.** For any rational number $t > 0$, the class of t-tough split graphs can be recognized in polynomial time.

Let t be an arbitrary positive rational number and now consider the following variants of the problem t-TOUGH.

EXACT-t-TOUGH
Instance: a graph G.
Question: is it true that $\tau(G) = t$?

MIN-t-TOUGH
Instance: a graph G.
Question: is it true that G is minimally t-tough?

Since extremal problems usually seem not to belong to NP $\cup$ coNP, the complexity class called DP was introduced by Papadimitriou and Yannakakis in [26].

Definition 2.12. A language L is in the class *DP* if there exist two languages $L_1 \in$ NP and $L_2 \in$ coNP such that $L = L_1 \cap L_2$.

A language is called *DP-hard* if all problems in DP can be reduced to it in polynomial time. A language is *DP-complete* if it is in DP and it is DP-hard.

It should be emphasized that DP $\neq$ NP $\cap$ coNP if NP $\neq$ coNP. Moreover, NP $\cup$ coNP $\subseteq$ DP.

Here we list some DP-complete problems that we use later for reduction.

EXACTINDEPENDENCENUMBER
Instance: a graph G and a positive integer k.
Question: is it true that $\alpha(G) = k$?

Note that, unlike t in the problem t-TOUGH, in this problem k is part of the input. The DP-completeness of EXACTINDEPENDENCENUMBER is a straightforward consequence of the following theorem.

Theorem 2.13 (Papadimitriou, Yannakakis, [26])**.** The following problem is DP-complete.

EXACTCLIQUE
Instance: a graph G and a positive integer k.
Question: is it true that the largest clique of G has size exactly k?

Corollary 2.14. The problem EXACTINDEPENDENCENUMBER is DP-complete.

α-CRITICAL
Instance: a graph G and a positive integer k.
Question: is it true that $\alpha(G) < k$, but $\alpha(G-e) \geq k$ for any edge $e \in E(G)$?

The DP-completeness of the problem α-CRITICAL is a trivial consequence of the following theorem.

Theorem 2.15 (Papadimitriou, Wolfe, [25])**.** The following problem is DP-complete.

CRITICALCLIQUE
Instance: a graph G and a positive integer k.
Question: is it true that G has no clique of size k, but adding any missing edge e to G, the resulting graph $G + e$ has a clique of size k?

Corollary 2.16. The problem α-CRITICAL is DP-complete.

2.5 α-critical graphs

Definition 2.17. A graph G is said to be α-critical if $\alpha(G-e) > \alpha(G)$ holds for any $e \in E(G)$.

Now we cite some results on α-critical graphs.

Proposition 2.18 (Problem 12 of §8 in [22])**.** If G is an α-critical graph without isolated points, then every point is contained in at least one maximum independent vertex set.

Lemma 2.19 (Problem 14 of §8 in [22])**.** If we replace a vertex of an α-critical graph with a clique, and connect every neighbor of the original vertex with every vertex in the clique, then the resulting graph is still α-critical.

Lemma 2.20 ([23]). Let G be an α-critical graph and w an arbitrary vertex of degree at least 2. Split w into two vertices y and z, each of degree at least 1, add a new vertex x to the graph and connect it to both y and z. Then the resulting graph G' is α-critical, and $\alpha(G') = \alpha(G) + 1$.

For one of our proofs we also need the following observation, which is a straightforward consequence of Corollary 2.16 and Lemmas 2.19 and 2.20.

Proposition 2.21. For any positive integers l and m, the following variant of the problem α-CRITICAL is DP-complete.

Instance: an l-connected graph G and a positive integer k that is divisible by m.
Question: is it true that $\alpha(G) < k$, but $\alpha(G - e) \geq k$ for any edge $e \in E(G)$?

Chapter 3

On the minimum degree of minimally 1-tough graphs

It follows directly from the definition that every t-tough noncomplete graph is $2t$-connected, implying $\kappa(G) \geq 2\tau(G)$ for noncomplete graphs (where t is a nonnegative real number). Therefore, the minimum degree of any t-tough noncomplete graph is at least $\lceil 2t \rceil$ for any positive real number t.

The following conjecture is motivated by a theorem of Mader [24] stating that every minimally k-connected graph has a vertex of degree k (where k is a positive integer).

Conjecture 3.1 (Kriesell [18])**.** Every minimally 1-tough graph has a vertex of degree 2.

This conjecture can be naturally generalized to any positive real number t.

Conjecture 3.2 (Generalized Kriesell Conjecture)**.** Every minimally t-tough graph has a vertex of degree $\lceil 2t \rceil$.

Since a minimally tough graph is not necessarily minimally connected (see Figure 3.1), Conjecture 3.2 does not follow from Mader's theorem directly.

Clearly, if a graph is Hamiltonian, then it must be 1-tough. However, not every 1-tough graph contains a Hamiltonian cycle: a well-known counterexample is the Petersen graph. On the other hand, Chvátal conjectured that there exists a positive real number t_0 such that every t_0-tough graph is Hamiltonian [13]. This conjecture is still open, but it is known that, if exists, t_0 must be at least $9/4$, see [7].

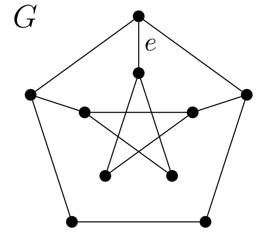

Figure 3.1: A minimally 1-tough but not minimally 2-connected graph. The graph $G - e$ is still 2-connected.

Since every Hamiltonian graph is 1-tough (and the toughness of K_3 is infinity), the only minimally 1-tough Hamiltonian graphs are cycles of length at least 4. Thus, Dirac's theorem provides a trivial upper bound on the minimum degree of minimally 1-tough graphs: since this theorem states that every graph on n vertices and with minimum degree at least $n/2$ contains a Hamiltonian cycle [14], the minimum degree of every minimally 1-tough graph is less than $n/2$, except for the cycle of length 4.

Here we improve this upper bound.

Theorem 3.3 (Katona, Soltész, Varga, [3])**.** Every minimally 1-tough graph on n vertices has a vertex of degree at most $n/3 + 1$.

3.1 Auxiliary results

First, we cite a theorem.

Theorem 3.4 (Häggkvist, Nicoghossian, [16])**.** Let G be a 2-connected graph on n vertices with

$$\delta(G) \geq \big(n + \kappa(G)\big)/3.$$

Then G is Hamiltonian.

Now we prove two lemmas.

Lemma 3.5. Let G be a minimally 1-tough graph on n vertices with $\delta(G) > n/3 + 1$. Let $e \in E(G)$ be an arbitrary edge and let $S = S(e)$ be a vertex set guaranteed by Proposition 2.4. Then $|S| > n/3$.

Proof. We can assume that S is of minimum size. Let $k = |S|$.

Since $\tau(G) = 1$, the graph G is 2-connected, i.e. e is not a bridge in G. So by Proposition 2.4,

$$\omega\big((G-e)-S\big) = |S| + 1 = k + 1.$$

Thus, at least one of the components of $(G-e)-S$ is of size at most $\lfloor (n-k)/(k+1) \rfloor$. If this component has size 1, then the vertex inside it has degree at most $k+1$ in G since all of the neighbors of this vertex are in S and if this vertex is one of the endpoints of e, then it has one more neighbor, namely, the other endpoint of e. Therefore,

$$\frac{n}{3} + 1 < \delta(G) \le k + 1,$$

which means that $k > n/3$. Otherwise, i.e. if this component has size at least 2, then there must exist a vertex in it which is not an endpoint of e, so in G this vertex has degree at most

$$\left\lfloor \frac{n-k}{k+1} \right\rfloor - 1 + k \le \frac{n-k}{k+1} - 1 + k = \frac{n + k^2 - k - 1}{k+1}.$$

Consider the function

$$f_n(k) = \frac{n + k^2 - k - 1}{k+1}.$$

Note that for any fixed n, the function f_n is monotone decreasing in k if $0 \le k \le \sqrt{n+1} - 1$ and monotone increasing if $\sqrt{n+1} - 1 < k \le n - 1$.

Now we show that if $k \le n/3$, then $\delta(G) \le n/3 + 1$.

Case 1: $2 \le k \le n/3$.

Since $f_n(k)$ is an upper bound on the minimum degree of G, it is enough to show that $f_n(k) \le \frac{n}{3} + 1$. The above mentioned property of the f_n implies that it is enough to show this for $k = 2$ and $k = n/3$.

$$f_n(2) = \frac{n+1}{3} < \frac{n}{3} + 1,$$

$$f_n\left(\frac{n}{3}\right) = \frac{n^2 + 6n - 9}{3n + 9} = \frac{(n+3)^2 - 18}{3(n+3)} < \frac{n+3}{3} = \frac{n}{3} + 1.$$

Case 2: $k = 1$.

Since $k = 1$, there exists a single vertex w whose removal from $G - e$ disconnects the graph. It is easy to see that every minimally 1-tough graph has at least 4 vertices. Thus w and one of the endpoints of e form a cutset in G, so $\kappa(G) \leq 2$. Since G is 1-tough, $\kappa(G) \geq 2$ holds. Thus $\kappa(G) = 2$. Since $\delta(G) > n/3 + 1$, Theorem 3.4 implies that G is Hamiltonian, but $G \neq C_n$, which contradicts the fact that G is minimally 1-tough. □

Lemma 3.6. If G is a minimally 1-tough graph with $\delta(G) > n/3 + 1$, then there are two vertices $a, b \in V(G)$ connected by an edge $f \in E(G)$ such that their open neighborhood (i.e., the set of vertices adjacent to a or b excluding a and b) has size more than $2n/3 - 1$.

Proof. Lemma 3.5 implies that $k(e) > n/3$ for all $e \in E(G)$. Let us fix an arbitrary edge $e \in E(G)$, and let $x = k - n/3$. It is easy to see that $0 < x < n/6$, because removing at least $n/2$ vertices does not leave enough components. Let $B = S(e)$ be a vertex set guaranteed by Proposition 2.4 and let A denote the set of the vertices of $G - B$. Then $|A| = 2n/3 - x$, and $|B| = n/3 + x$, and by the choice of B, the number of components of $G - B$ is also $n/3 + x$.

Our strategy is to prove that there exists a vertex $b \in B$ having at least $n/3 + 1$ neighbors in A and among these neighbors there exists a vertex a contained by a component of size at most 2 after the removal of B, see Figure 3.2. Since a has more than $n/3 - 1$ neighbors in $B \setminus \{b\}$ and b has at least $n/3$ neighbors in $A \setminus \{a\}$, their open neighborhood has size more than

$$\frac{n}{3} - 1 + \frac{n}{3} = \frac{2n}{3} - 1.$$

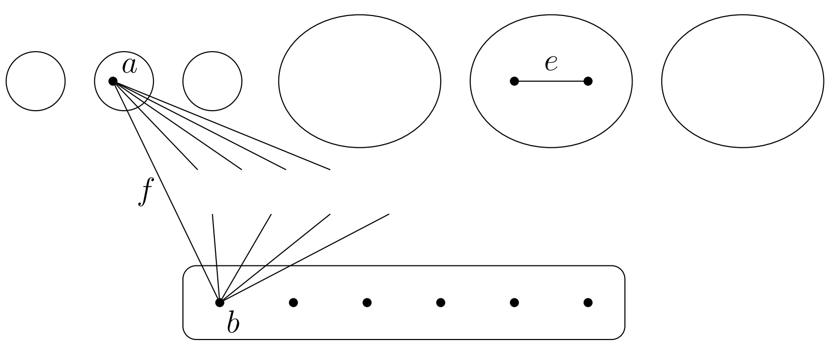

Figure 3.2: Finding an edge f for which $G - f$ is 1-tough.

So suppose to the contrary that there exist no such vertices a and b. Let $e(A,B)$ denote the number of edges between A and B. We give a lower and an upper bound on $e(A,B)$, then we show that the lower bound is greater than the upper bound, which leads us to a contradiction.

I. Lower bound:

$$e(A,B) > \frac{n^2}{9} + \frac{n}{3} + nx + x - 4x^2.$$

It is well-known that the number of the edges in a graph with n_0 vertices and k_0 components is at most $\binom{n_0-k_0+1}{2}$. Hence the number of the edges in A is at most

$$\binom{\left(\frac{2n}{3}-x\right)-\left(\frac{n}{3}+x\right)+1}{2} = \binom{\frac{n}{3}-2x+1}{2}.$$

Since every degree is more than $n/3+1$, the following lower bound can be given on $e(A,B)$.

$$\begin{aligned} e(A,B) &\geq \left(\frac{2n}{3}-x\right)\left(\frac{n}{3}+1\right) - 2\cdot\binom{\frac{n}{3}-2x+1}{2} = \\ &= \left(\frac{2n}{3}-x\right)\left(\frac{n}{3}+1\right) - \left(\frac{n}{3}-2x+1\right)\left(\frac{n}{3}-2x\right) = \\ &= \frac{n^2}{9}+\frac{n}{3}+nx+x-4x^2 \end{aligned}$$

II. Upper bound:

$$e(A,B) < \frac{n}{3}\left(\frac{n}{3}+1\right) + x\left(\frac{n}{2}-3x\right).$$

To prove this inequality, first we need the following claim.

Claim 3.7. After the removal of B, there are at least $n/6+2x$ components of size at most 2.

Proof. As we saw earlier, $G-B$ has $2n/3-x$ vertices and $n/3+x$ components. In every component there must be at least one vertex, so the other

$$\left(\frac{2n}{3}-x\right)-\left(\frac{n}{3}+x\right) = \frac{n}{3}-2x$$

vertices can create at most

$$\frac{1}{2} \cdot \left(\frac{n}{3} - 2x\right) = \frac{n}{6} - x$$

components of size at least 3. So there must be at least

$$\left(\frac{n}{3} + x\right) - \left(\frac{n}{6} - x\right) = \frac{n}{6} + 2x$$

components having size at most 2. □

Now we return to the proof of the upper bound on $e(A, B)$. Since $\delta(G) > n/3+1$, the vertices in the components of $G - B$ having size at most 2 have more than $n/3$ neighbors in B. By our assumption, each of these neighbors is connected to less than $n/3+1$ vertices in A. Consider the vertices of B which do not have neighbors in the components of $G - B$ of size at most 2. Clearly, there are less than x such vertices, and all of their neighbors in A lie in a component of size at least 3. So all these remaining less than x vertices in B can be adjacent to at most

$$\left(\frac{2n}{3} - x\right) - \left(\frac{n}{6} + 2x\right) = \frac{n}{2} - 3x$$

vertices in A.

Hence, there are more than $n/3$ vertices in B that have less than $n/3+1$ neighbors in A and the remaining less than x vertices in B have at most $n/2 - 3x$ neighbors in A, see Figure 3.3.

Now we show that $n/2 - 3x > n/3 + 1$. Intuitively this means that $e(A, B)$ is maximum if the components of size at most 2 together have as few neighbors as possible. This is an easy corollary of the following claim.

Claim 3.8. For the vertices of B, the average number of neighbors in A is more than $n/3 + 1$.

Proof. We have already seen that

$$e(A, B) > \frac{n^2}{9} + \frac{n}{3} + nx + x - 4x^2,$$

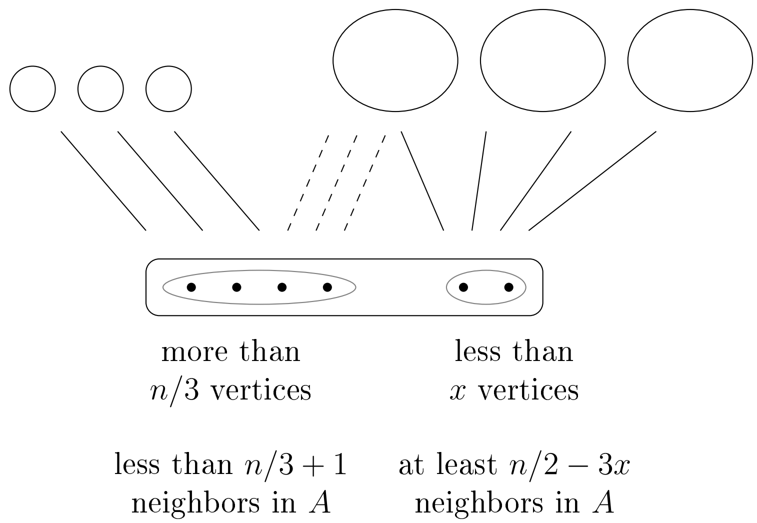

Figure 3.3: Giving an upper bound on $e(A,B)$.

so it is enough to show that

$$\frac{n^2}{9}+\frac{n}{3}+nx+x-4x^2>|B|\left(\frac{n}{3}+1\right)=\left(\frac{n}{3}+x\right)\left(\frac{n}{3}+1\right).$$

Transforming it into equivalent forms, we can see that this inequality holds.

$$\begin{aligned}\frac{n^2}{9}+\frac{n}{3}+nx+x-4x^2&>\frac{n^2}{9}+\frac{n}{3}+\frac{n}{3}x+x\\ \frac{2n}{3}x&>4x^2\\ \frac{n}{6}&>x\end{aligned}$$

□

So if $n/2-3x>n/3+1$ did not hold, then each vertex in B could be adjacent to at most $\frac{n}{3}+1$ vertices in A, which contradicts Claim 3.8. Hence

$$e(A,B)<\frac{n}{3}\cdot\left(\frac{n}{3}+1\right)+x\left(\frac{n}{2}-3x\right),$$

which completes the proof of the upper bound.

Clearly, the lower bound cannot be greater than the upper bound, so

$$\begin{aligned}\frac{n^2}{9}+\frac{n}{3}+nx+x-4x^2<\frac{n}{3}\cdot\left(\frac{n}{3}+1\right)+x\left(\frac{n}{2}-3x\right),\\ 0<x^2-\left(\frac{n}{2}+1\right)x,\\ 0<x\left[x-\left(\frac{n}{2}+1\right)\right],\end{aligned}$$

which contradicts the fact that $0<x<\frac{n}{6}$. Thus the proof of the lemma is complete. □

3.2 Upper bound on the minimum degree of minimally 1-tough graphs

Proof of Theorem 3.3. Suppose to the contrary that $\delta(G)>n/3+1$ and consider the edge $f=ab$ guaranteed by Lemma 3.6. Let $S=S(f)$ be a vertex set guaranteed by Proposition 2.4 and let $k=|S|$. Then Lemma 3.5 implies $k>n/3$. Obviously, the components of $(G-f)-S$ require

$$\omega\big((G-f)-S\big)=k+1>n/3+1$$

independent vertices: two of them can be a and b, but the rest of them cannot be adjacent either to a or to b, see Figure 3.4.

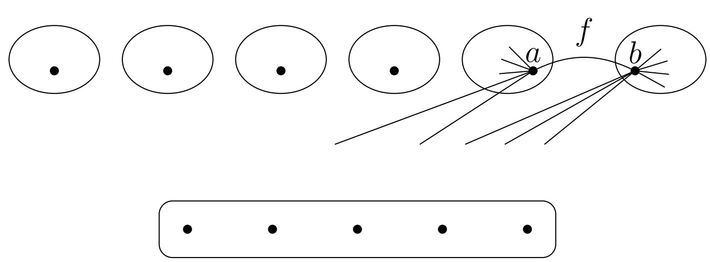

Figure 3.4: There are too many neighbors of a and b.

However, there are less than

$$n-\left(\frac{2n}{3}-1\right)=\frac{n}{3}+1<k+1$$

such vertices since a and b together have more than $2n/3-1$ different neighbors. So $G-f$ is 1-tough, which is a contradiction. $\square$

It is worth noting that Lemma 3.6 becomes trivial whenever the graph G is triangle-free. In Chapter 5 we show that supposing the graph is bipartite not only makes the whole proof easier, but in this case we can give an even better upper bound on the minimum degree.

Chapter 4

The complexity of recognizing minimally tough graphs

In this chapter, we study the problem of recognizing minimally tough graphs. The main result is the following.

Theorem 4.1 (Katona, Kovács, Varga, [2])**.** The problem MIN-t-TOUGH is DP-complete for any positive rational number t.

Note that since the toughness of any noncomplete graph is a rational number, there exist no minimally tough graphs with irrational toughness.

To prove the case $t \geq 1$, we introduce a new notion called weighted toughness.

Definition 4.2. Let t be a positive real number. Given a graph G and a positive weight function w on its vertices, we say that the graph G is *weighted t-tough* with respect to the weight function w if

$$\omega(G - S) \leq \frac{w(S)}{t}$$

holds for any vertex set $S \subseteq V(G)$ whose removal disconnects the graph, where

$$w(S) = \sum_{v \in S} w(v).$$

The *weighted toughness* of a noncomplete graph (with respect to the weight function w) is the largest t for which the graph is weighted t-tough, and we define the weighted toughness of complete graphs (with respect to w) to be infinity.

Note that the weighted toughness of a connected graph with respect to the weight function that assigns 1 to every vertex is the toughness of the graph.

4.1 Auxiliary results

Proposition 4.3. Let G be a connected noncomplete graph on n vertices. Then $\tau(G)$ is a positive rational number, and if $\tau(G) = a/b$, where a, b are relatively prime positive integers, then $1 \leq a, b \leq n-1$.

Proof. By definition,

$$\tau(G) = \min_{\substack{S \subseteq V(G) \\ \omega(G-S) \geq 2}} \frac{|S|}{\omega(G-S)}$$

for a noncomplete graph G. Since G is connected and noncomplete,

$$1 \leq |S| \leq n-2$$

for every $S \subseteq V(G)$ with $\omega(G-S) \geq 2$. Obviously, $\omega(G-S) \geq 2$, and since G is connected, $\omega(G-S) \leq n-1$. □

Corollary 4.4. Let G and H be two connected noncomplete graphs on n vertices. If $\tau(G) \neq \tau(H)$, then

$$\big|\tau(G) - \tau(H)\big| > \frac{1}{n^2}.$$

Proof. Let a, b and a', b' be two pairs of relative prime positive integers such that $\tau(G) = a/b$ and $\tau(H) = a'/b'$. Proposition 4.3 implies that $1 \leq a, b, a', b' \leq n-1$. Since $\tau(G) \neq \tau(H)$,

$$\big|\tau(G) - \tau(H)\big| = \left|\frac{a}{b} - \frac{a'}{b'}\right| = \left|\frac{ab' - a'b}{bb'}\right| > \frac{1}{n^2}.$$

□

Proposition 4.5. For every positive rational number t, the problem Min-t-Tough belongs to DP.

Proof. For any positive rational number t,

$$\begin{aligned}\textsc{Min-}t\textsc{-Tough} &= \big\{G \text{ graph} \bigm| \tau(G) = t \text{ and } \tau(G-e) < t \text{ for all } e \in E(G)\big\}\\ &= \big\{G \text{ graph} \bigm| \tau(G) \geq t\big\} \cap \big\{G \text{ graph} \bigm| \tau(G) \leq t\big\}\\ &\quad \cap \big\{G \text{ graph} \bigm| \tau(G-e) < t \text{ for all } e \in E(G)\big\}.\end{aligned}$$

Let

$$L_{1,1} = \big\{G \text{ graph} \bigm| \tau(G-e) < t \text{ for all } e \in E(G)\big\},$$
$$L_{1,2} = \big\{G \text{ graph} \bigm| \tau(G) \leq t\big\}$$

and

$$L_2 = \big\{G \text{ graph} \bigm| \tau(G) \geq t\big\}.$$

Notice that $L_2 = t$-TOUGH and it is known to be in coNP: if a graph G is not t-tough, then a witness is a vertex set $S \subseteq V(G)$ whose removal disconnects G and leaves more than $|S|/t$ components. Similarly, $L_{1,1} \in$ NP, since a witness is a set of vertex sets $\big\{S_e \subseteq V(G) \bigm| e \in E(G)\big\}$, where for any $e \in E(G)$ the removal of S_e disconnects $G - e$ and leaves more than $|S_e|/t$ components.

Now we show that $L_{1,2} \in$ NP, i.e. we can express $L_{1,2}$ in the form

$$L_{1,2} = \big\{G \text{ graph} \bigm| \tau(G) < t + \varepsilon\big\},$$

which is the complement of a language belonging to coNP. Let G be an arbitrary graph on n vertices. If G is disconnected, then $\tau(G) = 0$, and if G is complete, then $\tau(G) = \infty$, so in both cases $\tau(G) \leq t$ if and only if $\tau(G) < t + \varepsilon$ for any positive number ε. If G is connected and noncomplete, then from Corollary 4.4 it follows that $\tau(G) \leq t$ if and only if $\tau(G) < t + 1/n^2$. Therefore

$$L_{1,2} = \big\{G \text{ graph} \bigm| \tau(G) \leq t\big\} = \left\{G \text{ graph} \,\middle|\, \tau(G) < t + \frac{1}{|V(G)|^2}\right\},$$

so $L_{1,2} \in$ NP.

Since $L_{1,1} \cap L_{1,2} \in$ NP and $L_2 \in$ coNP and MIN-t-TOUGH $= (L_{1,1} \cap L_{1,2}) \cap L_2$, we can conclude that MIN-t-TOUGH $\in$ DP. □

4.2 On some special cases of Theorem 4.1

This section aims to highlight the key steps of the proof of Theorem 4.1 by considering some simpler cases of it. In the view of this intention, technical details are omitted here.

Let $n \geq 2$ be an integer and let G be a complete graph of size n on the vertex set $\{v_1, \ldots, v_n\}$. Add the vertices $u_1, \ldots, u_n$ and w to G, and for all $i \in [n]$ connect v_i and u_i, and also u_i and w, and let G' denote the obtained graph. (For an example see Figure A.1 in the Appendix.) It is easy to see that G' is a minimally 1-tough graph, and it is due to the fact that complete graphs are α-critical. This plain construction inspires all the others proposed in this paper. This construction can be generalized for α-critical graphs to obtain a minimally 1-tough graph. (See Figure A.2.) The construction for minimally integer-tough graphs can be seen as a "blow-up" of the minimally 1-tough construction. (See Figure A.3.) These constructions are described in details in the following subsection.

4.2.1 On the case of minimally t-tough graphs where t is a positive integer

Let t, k and $n \geq t+1$ be positive integers, let G be an arbitrary $\lceil (t+1)/2 \rceil$-connected graph on the vertices $v_1, \ldots, v_n$, and let $G'_{t,k}$ be defined as follows. For all $i \in [n]$ and $j \in [k]$ let

$$V_{i,j} = \left\{ v_{i,j,l} \;\middle|\; l \in [t] \right\}.$$

For all $i \in [n]$ let

$$V_i = \bigcup_{j \in [k]} V_{i,j}$$

and place a complete graph on its vertices. For all $i_1, i_2 \in [n]$ if $v_{i_1} v_{i_2} \in E(G)$, then place a complete bipartite graph on $(V_{i_1}; V_{i_2})$. For all $i \in [n]$ and $j \in [k]$ add the vertex set

$$U_{i,j} = \left\{ u_{i,j,l} \;\middle|\; l \in [t] \right\}$$

to the graph and place a complete graph on the vertices of $U_{i,j}$. For all $i \in [n], j \in [k], l \in [t]$ connect $v_{i,j,l}$ to $u_{i,j,l}$. For all $j \in [k]$ add the vertex set

$$W_j = \{w_{j,1}, \ldots, w_{j,t}\}$$

to the graph and for all $i \in [n]$ place a complete bipartite graph on $(U_{i,j}; W_j)$. Let

$$V = \bigcup_{i=1}^{n} V_i, \qquad U = \bigcup_{i=1}^{n} \bigcup_{j=1}^{k} U_{i,j}, \qquad W = \bigcup_{j=1}^{k} W_j.$$

See Figure 4.1. (For examples see Figures A.2 and A.3.)

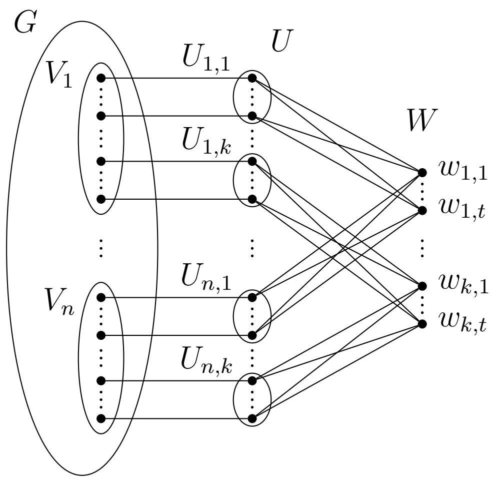

Figure 4.1: The graph $G'_{t,k}$, when t is a positive integer.

Claim 4.6. Let G be an arbitrary $\lceil (t+1)/2 \rceil$-connected graph. Then G is α-critical with $\alpha(G) = k$ if and only if $G'_{t,k}$ is minimally t-tough.

Proof. The cases $t = 1$ and $t \geq 2$ should be handled separately, but since the main steps of the proofs are similar, only the (easier) case $t = 1$ is presented here.

The proof of the following lemma is omitted now. (In Section 4.4 a similar lemma is proved, but for a more complex graph, see Lemma 4.19.)

Lemma 4.7. If $\alpha(G) \leq k$, then $\tau(G'_{1,k}) = 1$.

Accepting this lemma, all we have left to show is that

– if G is α-critical with $\alpha(G) = k$, then $\tau(G'_{1,k} - e) < 1$ holds for any $e \in E(G)$,

– if $\alpha(G) > k$, then $\tau(G'_{1,k}) < 1$, and

– if either $\alpha(G) = k$ but the graph G is not α-critical or $\alpha(G) < k$, then there exists an edge $e \in E(G)$ for which $\tau(G'_{1,k} - e) = 1$.

Assume first that G is α-critical with $\alpha(G) = k$. Let $e \in E(G'_{1,k})$ be an arbitrary edge. If e is incident to one of the vertices of U, i.e., to a vertex of degree 2, then clearly $\tau(G'_{1,k} - e) < 1$. If e is not incident to any of the vertices of U, then it connects two vertices of V. By Lemma 2.19, the subgraph $G'_{1,k}[V]$ is α-critical, so in $G'_{1,k}[V] - e$ there exists an independent vertex set I of size $\alpha(G) + 1$. Let

$$S = (V \setminus I) \cup W.$$

Then it is easy to see that

$$|S| = |V| - 1 \qquad \text{and} \qquad \omega\big((G'_{1,k} - e) - S\big) = |V|$$

hold, so $\tau(G'_{1,k} - e) < 1$.

Now assume $\alpha(G) > k$. Then let I be an independent vertex set of size $\alpha(G)$ in $G'_{1,k}[V]$, and let

$$S = (V \setminus I) \cup W.$$

Then

$$|S| < |V| \qquad \text{and} \qquad \omega(G'_{1,k} - S) = |V|$$

hold, so $\tau(G'_{1,k}) < 1$.

Finally, assume that either $\alpha(G) = k$ but the graph G is not α-critical or $\alpha(G) < k$. Then there exists an edge $e \in E(G)$ such that $\alpha(G - e) \leq k$. By Lemma 4.7, the graph $(G-e)'_{1,k}$ is 1-tough, but we can obtain $(G-e)'_{1,k}$ from $G'_{1,k}$ by edge-deletion, which means that $G'_{1,k}$ is not minimally 1-tough. □

Corollary 4.8. For any positive integer t, the problem MIN-t-TOUGH is DP-complete.

Proof. In Proposition 4.5 we already proved that the problem MIN-t-TOUGH is in DP, and it follows from Claim 4.6 that we can reduce the variant of α-CRITICAL defined in Proposition 2.21 with the choice of $l = \lceil (t+1)/2 \rceil$ and $m = 1$ to it, but for this it should be also noted that $G'_{t,k}$ can be constructed from G in polynomial time. □

The above construction works only in the case when t is a positive integer for the simple reason that the sets $V_{i,j}$, $U_{i,j}$ and W_j consist of t vertices.

4.2.2 On the case of minimally $1/b$-tough graphs where $b \geq 2$ is an integer

Up to this point, we only handled the case when t is a positive integer. To prove Theorem 4.1 for the noninteger cases, we modify the previous constructions and here we illustrate these modifications with the following simple example.

Let $b \geq 2$ be an integer, let $t = 1/b$, let G be an arbitrary connected graph, and let G_t be defined as follows. Add $b-1$ independent vertices for each original vertex $v \in V(G)$ to the graph G, and connect them to v (see Figure 4.2). (For an example see Figure A.4.)

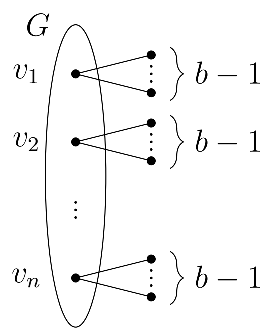

Figure 4.2: The graph G_t when $t = 1/b$, where $b \geq 2$ is an integer.

Claim 4.9. Let G be an arbitrary connected graph, $b \geq 2$ an integer and $t = 1/b$. Then G_t is minimally t-tough if and only if G is almost minimally 1-tough.

Similarly as before, we can conclude the following.

Corollary 4.10. For every integer $b \geq 2$, MIN-$1/b$-TOUGH is DP-complete.

In Section 4.5, this latter idea is extended to the case when $t \leq 1/2$ by "gluing" some other graph to the vertices of the original graph G. (See Figure A.5.) It is worth noting that in the case when $t = 1/b$ for some integer $b \geq 2$, the obtained graph in Section 4.5 is exactly the same as the graph G_t constructed here. After this "gluing", the vertices of G become cut-vertices in the obtained graph G_t, thus the toughness of G_t can be at most $1/2$. The plan for the cases when $t > 1/2$ is to perform this so called "gluing" by identifying not only one, but $\lceil 2t \rceil$ vertices of a smaller and a larger graph, where the larger graph resembles a minimally $\lceil t \rceil$-tough graph and the "gluing" procedure aims to decrease its toughness to the desired value t. In fact, in Sections 4.3 and 4.4 this larger graph is chosen to be a slight modification of $G'_{\lceil t \rceil,k}$.

4.3 Minimally t-tough graphs where $1/2 < t < 1$

Before proving Theorem 4.1 for any positive rational number $1/2 < t < 1$, we need some preparation. First, we construct some auxiliary graphs.

4.3.1 The auxiliary graph $H^{**}_{t,k}$ when $1/2 < t < 1$

Let t be a rational number such that $1/2 < t < 1$. Let a, b be relatively prime positive integers such that $t = a/b$. Let k be a positive integer, and let

$$W = \{w_1, \dots, w_{ak}\} \qquad \text{and} \qquad W' = \{w'_1, \dots, w'_{(b-1)k}\}.$$

Place a clique on the vertices of W and a complete bipartite graph on $(W; W')$. Obviously, the toughness of this complete split graph is $a/(b-1) > t$. Deleting an edge may decrease the toughness, and now we delete edges incident to W' until the toughness remains at least t but the deletion of any other such edge would result in a graph with toughness less than t. Let $H^*_{t,k}$ denote the obtained split graph. Then $\tau(H^*_{t,k}) \geq t$, and $\tau(H^*_{t,k} - e) < t$ for any edge $e \in E(H^*_{t,k})$ incident to W', i.e. there exists a vertex set $S = S(e) \subseteq W$ whose removal disconnects $H^*_{t,k} - e$ and

$$\omega\big((H^*_{t,k} - e) - S\big) > \frac{|S|}{t}.$$

Now delete all the edges induced by W, and let $H^{**}_{t,k}$ denote the obtained bipartite graph.

4.3.2 The auxiliary graph H''_t when $1/2 < t < 1$

Let t be a rational number such that $1/2 < t < 1$. Let a, b be relatively prime positive integers such that $t = a/b$ and let H_t be constructed as follows. Let

$$A = \{v_1, v_2, \dots, v_a\}, \quad B = \{u_1, u_2, \dots, u_b\}.$$

For any $i \in [a]$ and $j \in [b-1]$ connect v_i to u_j, and connect u_b to v_1 and v_a. (In other words, H_t can be obtained from the complete bipartite graph $K_{a,b}$ by deleting $a-2$ edges incident to one vertex of the color class of size b. See Figure 4.3.)

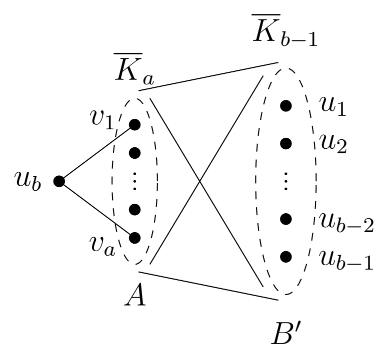

Figure 4.3: The graph H_t, when $1/2 < t < 1$.

Claim 4.11. Let t be a rational number such that $1/2 < t < 1$. Then $\tau(H_t) = t$.

Proof. Let $B' = B \setminus \{u_b\}$ and let S be an arbitrary cutset in H_t. Now we show that $\omega(H_t - S) \leq |S|/t$.

Case 1: $A \subseteq S$.
Then $|S| \geq a$ and $\omega(H_t - S) \leq b$. Since $t = a/b < 1$, it follows that

$$\omega(H_t - S) \leq b = \frac{a}{t} \leq \frac{|S|}{t}.$$

Case 2: $B' \subseteq S$.
If $u_b \in S$ as well, then $|S| \geq b$ and $\omega(H_t - S) \leq a$. Since $t = a/b < 1$, it follows that

$$\omega(H_t - S) \leq a = bt < \frac{b}{t} \leq \frac{|S|}{t}.$$

If $u_b \notin S$, then $|S| \geq b-1$ and $\omega(H_t - S) \leq a-1$. Since $t = a/b < 1$, it follows that

$$\omega(H_t - S) \leq a - 1 \leq \frac{b-1}{t} \leq \frac{|S|}{t}.$$

Case 3: $A \nsubseteq S$ and $B' \nsubseteq S$.

Then $\omega(H_t - S) \leq 2$, but since S is a cutset, $\omega(H_t - S) = 2$. Obviously, there is no cut-vertex in H_t, thus $|S| \geq 2$. Since $t < 1$, it follows that

$$\omega(H_t - S) = 2 < \frac{2}{t} \leq \frac{|S|}{t}.$$

Hence $\tau(H_t) \geq t$. On the other hand, the vertex set $S = A$ is a cutset in H_t with $|S| = a$ and $\omega(H_t - S) = b$, so $\tau(H_t) \leq t$.

Therefore, $\tau(H_t) = t$. □

By repeatedly deleting some edges of H_t, eventually we obtain a minimally t-tough graph, let us denote it with H'_t (i.e. if there exists an edge whose deletion does not decrease the toughness, then we delete it). Obviously, we could not delete the edges incident to u_b, so the vertex u_b still has degree 2. Let e denote the edge connecting v_1 and u_b and let $H''_t = H'_t - e$. Note that H''_t is a bipartite graph with color classes A and B.

4.3.3 The proof of Theorem 4.1 when $1/2 < t < 1$

Theorem 4.12 (Katona, Kovács, Varga, [2]). For any rational number t with $1/2 < t < 1$, the problem Min-t-Tough is DP-complete.

Proof. Let t be a rational number such that $1/2 < t < 1$. In Proposition 4.5 we already proved that the problem Min-t-Tough is in DP. To show that it is DP-hard, we reduce the variant of α-Critical definied in Proposition 2.21 with the choice of $l = 2$ and $m = 1$ to it.

Let a, b be relatively prime positive integers such that $t = a/b$, let G be an arbitrary 2-connected graph on the vertices $v_1, \dots, v_n$ and let $G_{t,k}$ be defined as follows. For all $i \in [n]$ let

$$V_i = \left\{ v_{i,j} \;\middle|\; i \in [n], j \in [ak] \right\}$$

and place a clique on the vertices of V_i. For all $i_1, i_2 \in [n]$ if $v_{i_1}v_{i_2} \in E(G)$, then place a complete bipartite graph on $(V_{i_1}; V_{i_2})$. (This subgraph is denoted by $\tilde{G}$ in Figure 4.4.) For all $i \in [n], j \in [ak]$ "glue" the graph H''_t to the vertex $v_{i,j}$ by identifying $v_{i,j}$ with the vertex v_1 of H''_t and let $H^{i,j}$ denote the (i,j)-th copy of H''_t and let $A^{i,j}$ denote the (i,j)-th copy of its color class

A, and let $v'_{i,j}$ and $u_{i,j}$ denote the (i,j)-th copies of the vertices v_a and u_b, respectively. Let

$$V = \bigcup_{i=1}^{n} V_i,$$

$$V' = \left\{ v'_{i,j} \;\middle|\; i \in [n], j \in [ak] \right\}$$

and

$$U = \left\{ u_{i,j} \;\middle|\; i \in [n], j \in [ak] \right\}.$$

Add the vertex sets

$$W = \left\{ w_j \;\middle|\; j \in [ak] \right\}$$

and

$$W' = \left\{ w'_1, \dots, w'_{(b-1)k} \right\}$$

to the graph and place the bipartite graph $H^{**}_{t,k}$ on $(W; W')$. For all $i \in [n]$ and $j \in [ak]$ connect w_j to $u_{i,j}$. See Figure 4.4. (For an example see Figure A.6.) Now k is part of the input of the problem α-CRITICAL, therefore the graph $H^{**}_{t,k}$ must be constructed in polynomial time and by Theorem 2.11, this can be done. On the other hand, t is not part of the input of the problem MIN-t-TOUGH, therefore the graph H''_t can be constructed in advance. Hence, $G_{t,k}$ can be constructed from G in polynomial time.

To show that G is α-critical with $\alpha(G) = k$ if and only if $G_{t,k}$ is minimally t-tough, first we prove the following lemma.

Lemma 4.13. Let G be a 2-connected graph with $\alpha(G) \leq k$. Then $G_{t,k}$ is t-tough.

Proof. Let $S \subseteq V(G_{t,k})$ be a cutset in $G_{t,k}$. We need to show that $\omega(G_{t,k} - S) \leq |S|/t$.

First, we show that the following assumption can be made for S.

(1) $U \cap S = \emptyset$.

Suppose that $u_{i,j} \in S$ for some $i \in [n], j \in [ak]$. If $v'_{i,j} \in S$, then after the removal of $v'_{i,j}$, the vertex $u_{i,j}$ has degree 1, so there is no need to remove it. Similarly, if $w_j \in S$, then we can also assume that $u_{i,j} \notin S$. If $v'_{i,j}, w_j \notin S$, then considering $S' = S \setminus \{u_{i,j}\}$ instead of S decreases the number of components only by one, meaning that if S' is a cutset

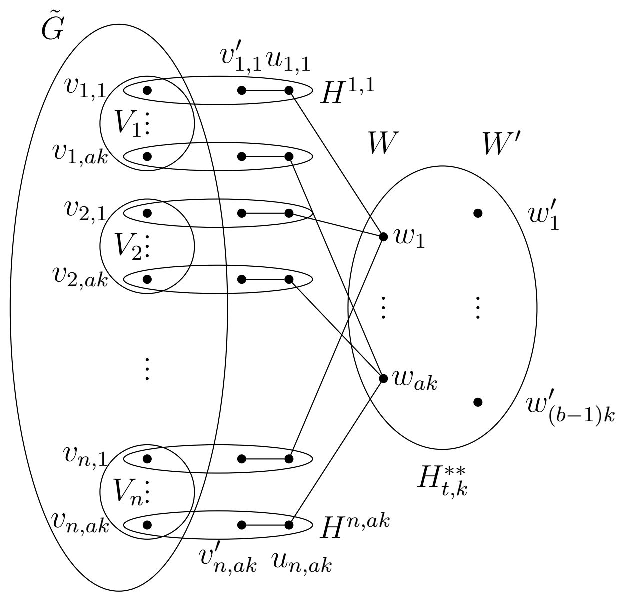

Figure 4.4: The graph $G_{t,k}$, when $1/2 < t < 1$.

in $G_{t,k}$, then it is enough to show that $\omega(G_{t,k} - S') \leq |S'|/t$ since it implies

$$\omega(G_{t,k} - S) = \omega(G_{t,k} - S') + 1 \leq \frac{|S'|}{t} + 1 = \frac{|S| - 1}{t} + 1 \leq \frac{|S|}{t},$$

where the last inequality is valid since $t < 1$. If S' is not a cutset in $G_{t,k}$, then $\omega(G_{t,k} - S) = 2$ and $|S| \geq 2$ since $u_{i,j}$ has degree 2 and is not a cut-vertex in $G_{t,k}$, i.e.

$$\omega(G_{t,k} - S) = 2 \leq |S| \leq \frac{|S|}{t},$$

where again the last inequality is valid since $t < 1$. This completes the validation of assumption (1).

Now there are two cases.

Case 1: $W \subseteq S$.

After the removal of W, the vertices of W' are isolated; therefore we can assume that $W' \cap S = \emptyset$.

To write up a formula for $|S|$ and $\omega(G_{t,k}-S)$, we need to introduce some notations. Let

$$C = \big\{(i,j) \in [n] \times [ak] \;\big|\; v_{i,j} \in V \cap S\big\},$$

$$c_{i,j} = \big|V(H^{i,j}) \cap S\big| - 1$$

for all $(i,j) \in C$, and

$$d_{i,j} = \big|V(H^{i,j}) \cap S\big|$$

for all $(i,j) \in \big([n] \times [ak]\big) \setminus C$. Finally, let

$$D = \big\{(i,j) \in \big([n] \times [ak]\big) \setminus C \;\big|\; d_{i,j} > 0\big\}.$$

Using these notations it is clear that

$$|S| = \sum_{(i,j)\in[n]\times[ak]} \big|V(H^{i,j}) \cap S\big| + |W| = |C| + \sum_{(i,j)\in C} c_{i,j} + \sum_{(i,j)\in D} d_{i,j} + ak.$$

By the assumption that $W \subseteq S$, in $G_{t,k} - S$ the $(b-1)k$ vertices of W' are isolated. Since $\alpha\big(G_{t,k}[V]\big) = \alpha(G)$, the removal of $V \cap S$ from $G_{t,k}[V]$ leaves at most $\alpha(G)$ components. By Claim 4.11 and Proposition 2.2, for any $(i,j) \in C$ the removal of $V(H^{i,j}) \cap S$ from $H^{i,j}$ leaves at most $(c_{i,j}+1)/t$ components. By Proposition 2.2, for any $(i,j) \in D$ the removal of $V(H^{i,j}) \cap S$ from $H^{i,j}$ leaves at most $d_{i,j}/t+1$ components, but the component of $v_{i,j}$ has been already counted. Hence

$$\begin{aligned}\omega(G_{t,k}-S) &\le (b-1)k + \alpha(G) + \sum_{(i,j)\in C} \frac{c_{i,j}+1}{t} + \sum_{(i,j)\in D} \frac{d_{i,j}}{t} \\ &\le bk + \frac{|C| + \sum_{(i,j)\in C} c_{i,j} + \sum_{(i,j)\in D} d_{i,j}}{t} = \frac{|S|}{t},\end{aligned}$$

using that $\alpha(G) \le k$.

Case 2: $W \nsubseteq S$.
Assume that $w_{j_0} \notin S$ for some $j_0 \in [ak]$. In this case, using assumption (1), we can also assume the following.

(2) There exists at most one $i \in [n]$ for which $v_{i,j_0} \in S$.

Suppose that $v_{i_1,j_0}, v_{i_2,j_0} \in S$ for some $i_1, i_2 \in [n]$. By assumption (1), the component of w_{j_0} contains all of the vertices $u_{1,j_0}, u_{2,j_0}, \ldots, u_{n,j_0}$.

Now considering the cutset $S' = S \cup \{w_{j_0}\}$ instead of S increases the number of components by at least two: it disconnects both u_{i_1,j_0} and u_{i_2,j_0} from the vertices $\{u_{i,j_0} \mid i \in [n] \setminus \{i_1, i_2\}\}$, and it also disconnects u_{i_1,j_0} from u_{i_2,j_0} (and of course it can also disconnect other vertices of $\{u_{i,j_0} \mid i \in [n]\}$ from each other). Then it is enough to show that $\omega(G_{t,k} - S') \le |S'|/t$ since it implies

$$\omega(G_{t,k} - S) \le \omega(G_{t,k} - S') - 2 \le \frac{|S'|}{t} - 2 = \frac{|S| + 1}{t} - 2 < \frac{|S|}{t},$$

where the last inequality is valid since $t > 1/2$. Proceeding further, we can obtain a cutset S^* for which $W \subseteq S^*$ holds; and such sets were already handled in Case 1.

(3) $(G_{t,k} - S)[V]$ is connected.

By assumption (2), there exists at most one $i \in [n]$ for which $V_i \subseteq S$. Since G is 2-connected, this implies that $(G_{t,k} - S)[V]$ is connected.

(4) There exists at most one $i \in [n]$ for which v_{i,j_0} and u_{i,j_0} belong to different components in $G_{t,k} - S$.

Suppose that v_{i_1,j_0}, u_{i_1,j_0} belong to different components in $G_{t,k} - S$, and so do v_{i_2,j_0}, u_{i_2,j_0} for some $i_1, i_2 \in [n]$. Similarly as in the proof of assumption (2), considering the cutset $S' = S \cup \{w_{j_0}\}$ instead of S increases the number of components by at least two, so it is enough to show that $\omega(G_{t,k} - S') \le |S'|/t$.

(5) In $G_{t,k} - S$ all the remaining vertices of $\{v_{i,j_0}, u_{i,j_0} \mid i \in [n]\}$ belong to the component of w_{j_0}.

It follows directly from assumptions (1), (2) and (3).

(6) In $G_{t,k} - S$ all the remaining vertices of V belong to the component of w_{j_0}.

It follows directly from assumptions (3) and (5).

(7) In $G_{t,k} - S$ all the remaining vertices of $V \cup W$ belong to the same component.

It follows directly from assumptions (5) and (6).

By assumption (7), in $G_{t,k} - S$ there is a component containing all the remaining vertices of $V \cup W$, and every other component is either an isolated vertex of W' (since $G_{t,k}[W \cup W']$ is a bipartite graph) or a component of $H^{i,j} - \big(V(H^{i,j}) \cap S\big)$ for some $i \in [n], j \in [ak]$. Hence we can also assume the following.

(8) $W' \cap S = \emptyset$.

By assumption (5) and Proposition 2.2 and the properties of $H^{**}_{t,k}$, the removal of $W \cap S$ from $H^{**}_{t,k}$ leaves at most $|W \cap S|/t$ components, but the component of w_{j_0} has been already counted.

Using the previous notations,

$$|S| = |C| + \sum_{(i,j)\in C} c_{i,j} + \sum_{(i,j)\in D} d_{i,j} + |W \cap S|$$

and

$$\begin{aligned}\omega(G_{t,k} - S) &\le 1 + \sum_{(i,j)\in C} \frac{c_{i,j}+1}{t} + \sum_{(i,j)\in D} \frac{d_{i,j}}{t} + \left(\frac{|W \cap S|}{t} - 1\right)\\ &= \frac{|C| + \sum_{(i,j)\in C} c_{i,j} + \sum_{(i,j)\in D} d_{i,j}}{t} + \frac{|W \cap S|}{t} = \frac{|S|}{t}.\end{aligned}$$

This means that $\tau(G_{t,k}) \ge t$. $\square$

Now we return to the proof of Theorem 4.12 and we show that G is α-critical with $\alpha(G) = k$ if and only if $G_{t,k}$ is minimally t-tough.

Let us assume that G is α-critical with $\alpha(G) = k$. By Lemma 4.13, the graph $G_{t,k}$ is t-tough, i.e. $\tau(G_{t,k}) \ge t$.

Let I be an independent vertex set of size $\alpha(G)$ in $G_{t,k}[V]$.

Recall the definition of $A^{i,j}$ from the beginning of the proof: it is the color class A in the corresponding copy of H''_t. Let

$$J = \big\{(i,j) \in [n] \times [ak] \;\big|\; v_{i,j} \in I\big\}$$

and

$$S = \left(\bigcup_{(i,j)\notin J} A^{i,j}\right) \cup W.$$

Then S is a cutset in $G_{t,k}$ with

$$|S| = a\big(|V| - \alpha(G)\big) + ak = a|V|$$

and

$$\omega(G_{t,k} - S) = \alpha(G) + b\big(|V| - \alpha(G)\big) + (b-1)k = b|V| = \frac{|S|}{t},$$

so $\tau(G_{t,k}) \le t$.

Therefore, $\tau(G_{t,k}) = t$.

Let $e \in E(G_{t,k})$ be an arbitrary edge. We need to show that $\tau(G_{t,k} - e) < t$. Now we have four cases.

Case 1: e has an endpoint in U.

Then this endpoint has degree 2, so $\tau(G_{t,k} - e) \le 1/2 < t$.

Case 2: e has an endpoint in W'.

By the properties of $H^*_{t,k}$, there exists a cutset $S \subseteq W$ in $H^*_{t,k} - e$ for which

$$\omega\big((H^*_{t,k} - e) - S\big) > \frac{|S|}{t}.$$

Note that S is also a cutset in $G_{t,k} - e$ and

$$\omega\big((G_{t,k} - e) - S\big) > \frac{|S|}{t},$$

so $\tau(G_{t,k} - e) < t$.

Case 3: e is induced by H^{i_0,j_0} for some $i_0 \in [n], j_0 \in [ak]$.

By Proposition 2.4, there exists a vertex set $S \subseteq V(H'_t)$ for which

$$\omega\big((H'_t - e) - S\big) > \frac{|S|}{t}.$$

Consider the (i_0, j_0)-th copy of the vertex set S in $G_{t,k} - e$; let us denote it with S_{i_0,j_0}. If $v_{i_0,j_0} \in S_{i_0,j_0}$, then S_{i_0,j_0} is a cutset in $G_{t,k} - e$ and

$$\omega\big((G_{t,k} - e) - S_{i_0,j_0}\big) = \omega\big((H'_t - e) - S\big) > \frac{|S|}{t},$$

so $\tau(G_{t,k}-e)<t$. Now assume that $v_{i_0,j_0}\notin S_{i_0,j_0}$. Let I be an independent vertex set of size $\alpha(G)$ in $G_{t,k}[V]$ that contains v_{i_0,j_0} (by Proposition 2.18, such an independent vertex set exists). Let

$$J=\big\{(i,j)\in[n]\times[ak]\;\big|\;v_{i,j}\in I\big\}$$

and

$$S'=S_{i_0,j_0}\cup\left(\bigcup_{(i,j)\notin J}A^{i,j}\right)\cup W.$$

Then S' is a cutset in $G_{t,k}-e$ with

$$|S'|=|S|+a\big(|V|-\alpha(G)\big)+ak=|S|+a|V|$$

and

$$\omega\big((G_{t,k}-e)-S'\big)>\frac{|S|}{t}+\alpha(G)+b\big(|V|-\alpha(G)\big)+(b-1)k=\frac{|S|}{t}+b|V|=\frac{|S'|}{t},$$

so $\tau(G_{t,k}-e)<t$.

Case 4: e connects two vertices of V.

By Lemma 2.19, the graph $G_{t,k}[V]$ is α-critical, so in $(G_{t,k}-e)[V]$ there exists an independent vertex set I of size $\alpha(G)+1$. Let

$$J=\big\{(i,j)\in[n]\times[ak]\;\big|\;v_{i,j}\in I\big\}$$

and

$$S=\left(\bigcup_{(i,j)\notin J}A^{i,j}\right)\cup W.$$

Then S is a cutset in $G_{t,k}-e$ with

$$|S|=a\big(|V|-\alpha(G)-1\big)+ak=a|V|-a$$

and

$$\omega\big((G_{t,k}-e)-S\big)=\alpha(G)+1+b\big(|V|-\alpha(G)-1\big)+(b-1)k=b|V|-b+1>\frac{|S|}{t},$$

so $\tau(G_{t,k}-e)<t$.

Therefore, if G is α-critical with $\alpha(G) = k$, then $G_{t,k}$ is minimally t-tough.

Now let us assume that G is not α-critical with $\alpha(G) = k$, i.e. either $\alpha(G) \neq k$ or even though $\alpha(G) = k$, the graph G is not α-critical.

Case I: $\alpha(G) > k$.
Let I be an independent vertex set of size $\alpha(G)$ in $G_{t,k}[V]$ and let

$$J = \big\{(i,j) \in [n] \times [ak] \;\big|\; v_{i,j} \in I\big\}$$

and

$$S = \left(\bigcup_{(i,j) \notin J} A^{i,j} \right) \cup W.$$

Then S is a cutset in $G_{t,k} - e$ with

$$|S| = a\big(|V| - \alpha(G)\big) + ak = a|V| - a\big(\alpha(G) - k\big)$$

and

$$\begin{aligned}\omega(G_{t,k} - S) = \alpha(G) + b\big(|V| - \alpha(G)\big) + (b-1)k &= b|V| - (b-1)\big(\alpha(G) - k\big) \\ &> b|V| - b\big(\alpha(G) - k\big) = \frac{|S|}{t},\end{aligned}$$

so $\tau(G_{t,k}) < t$, which means that $G_{t,k}$ is not minimally t-tough.

Case II: $\alpha(G) \leq k$.
Since G is not α-critical with $\alpha(G) = k$, there exists an edge $e \in E(G)$ such that $\alpha(G - e) \leq k$. By Lemma 4.13, the graph $(G - e)_{t,k}$ is t-tough, but it can be obtained from $G_{t,k}$ by edge-deletion, which means that $G_{t,k}$ is not minimally t-tough. $\square$

4.4 Minimally t-tough graphs where $t \geq 1$

This whole section resembles the previous one in structure. However, it requires some additional ideas that make the proofs more complicated. First, again, we construct some auxiliary graphs.

Let $t \geq 1$ be a rational number. It is easy to see that either $\lceil 2t \rceil = 2\lceil t \rceil$ or $\lceil 2t \rceil = 2\lceil t \rceil - 1$. Let $T = \lceil t \rceil$,

$$T' = \lceil 2t \rceil - \lceil t \rceil = \begin{cases} T & \text{if } \lceil 2t \rceil = 2\lceil t \rceil, \\ T-1 & \text{if } \lceil 2t \rceil = 2\lceil t \rceil - 1, \end{cases}$$

and

$$M = \left\lceil \frac{2\lceil t \rceil}{\lceil 2t \rceil} \right\rceil = \begin{cases} 1 & \text{if } \lceil 2t \rceil = 2\lceil t \rceil, \\ 2 & \text{if } \lceil 2t \rceil = 2\lceil t \rceil - 1. \end{cases}$$

Let a, b be the smallest positive integers such that $b \geq 3$ and $t = a/b$.

4.4.1 The auxiliary graph $H^{**}_{t,k}$ when $t \geq 1$

Let k be a positive integer that is divisible by a. Note that in this case

$$\left(\frac{MT'}{t} - 1\right) k = \begin{cases} \frac{Tbk}{a} - k & \text{if } \lceil 2t \rceil = 2\lceil t \rceil, \\ \frac{2(T-1)bk}{a} - k & \text{if } \lceil 2t \rceil = 2\lceil t \rceil - 1 \end{cases}$$

is a positive integer. Let

$$W = \left\{ w_{j,l,m} \;\middle|\; j \in [k], l \in [T'], m \in M \right\}$$

and

$$W' = \left\{ w'_1, \dots, w'_{(MT'/t-1)k} \right\}.$$

Place a clique on the vertices of W and a complete bipartite graph on $(W; W')$. Obviously, the toughness of this complete split graph is

$$\frac{kMT'}{(MT'/t-1)k} = \frac{1}{\frac{1}{t} - \frac{1}{MT'}} > t.$$

Deleting an edge may decrease the toughness, and now we delete edges incident to W' until the toughness remains at least t but the deletion of any other such edge would result in a graph with toughness less than t. Let $H^*_{t,k}$ denote the obtained split graph. Then $\tau(H^*_{t,k}) \geq t$, and $\tau(H^*_{t,k} - e) < t$ for any edge $e \in E(H^*_{t,k})$ incident to W', i.e. there exists a vertex set $S = S(e) \subseteq W$ whose removal disconnects $H^*_{t,k} - e$ and

$$\omega\big((H^*_{t,k} - e) - S\big) > \frac{|S|}{t}.$$

Now delete all the edges induced by W, and let $H^{**}_{t,k}$ denote the obtained bipartite graph.

4.4.2 The auxiliary graph $H''_{t,k}$ when $t \geq 1$

Let H_t be constructed as follows. Let

$$V'_1 = \{v'_1, \dots, v'_T\}, \qquad V'_2 = \{v'_{T+1}, \dots, v'_{2T}\}, \qquad V'_3 = \{v'_{2T+1}, \dots, v'_{aT}\},$$
$$V'' = \{v''_1, \dots, v''_T\},$$
$$U'_1 = \{u'_1, \dots, u'_T\}, \qquad U'_2 = \{u'_{T+1}, \dots, u'_{2T}\}, \qquad U'_3 = \{u'_{2T+1}, \dots, u'_{bT-1}\},$$
$$U'' = \{u''_1, \dots, u''_{T'}\},$$

and

$$U''_1 = \{u''_1, \dots, u''_T\}.$$

Place a clique on the vertices of V'_1, V'_2, V'_3, and U''. For all $l \in [T]$ connect v''_l to v'_l and to u'_l, and connect v'_{T+l} to u'_{T+l}. Connect all the vertices of V'_3 to all the vertices of $V'_1 \cup V'' \cup U'_1 \cup U'_2$, and connect all the vertices of V'_2 to all the vertices of U''. Finally, add a new vertex x to the graph and connect it to all the vertices of $V'_1 \cup U''$. See Figure 4.5.

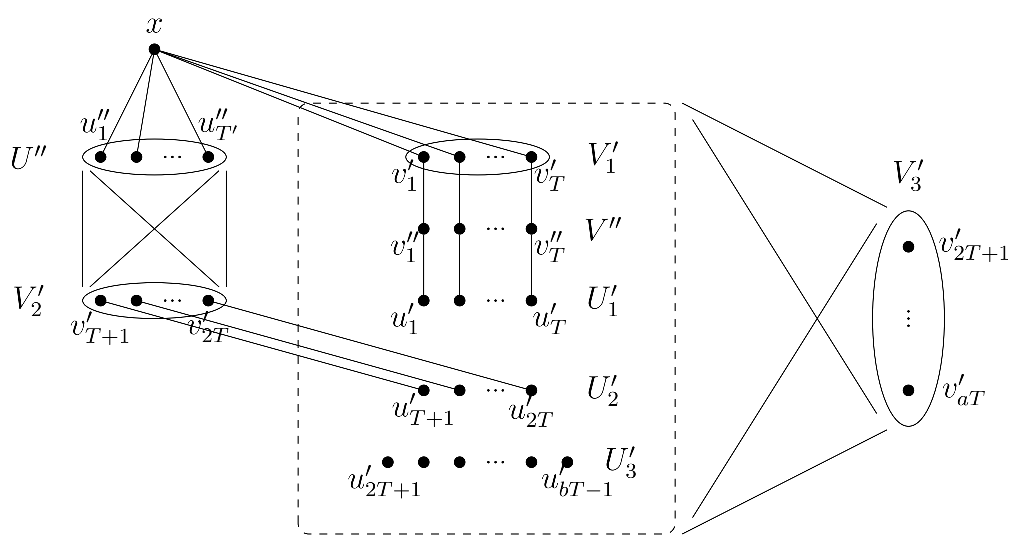

Figure 4.5: The graph H_t, when $t \geq 1$.

Claim 4.14. For any rational number $t \geq 1$, the graph H_t has weighted toughness t with respect to the weight function w that assigns weight 1 to all the vertices of H_t except for the vertex x, to which it assigns weight t.

Proof. Let S be an arbitrary cutset of H_t. We need to show that $\omega(H_t - S) \leq w(S)/t$.

We can assume that either $V_3' \cap S = \emptyset$ or $V_3' \subseteq S$ since removing only a proper subset of V_3' does not disconnect anything from the graph. Similarly, we can also assume that either $U'' \cap S = \emptyset$ or $U'' \subseteq S$.

Case 1: $V_3' \cap S = \emptyset$ *and* $U'' \cap S = \emptyset$.

Then $H_t - S$ has at most 2 components, and to obtain 2 components, the following must hold:

– $u'_{T+l} \in S$ or $v'_{T+l} \in S$ for all $l \in [T]$, and

– $x \in S$ or $V_1' \subseteq S$.

Hence $w(S) \geq T + t$ and

$$\omega(H_t - S) = 2 \leq \frac{T+t}{t} \leq \frac{w(S)}{t}.$$

Case 2: $V_3' \cap S = \emptyset$ *and* $U'' \subseteq S$.

Now we can assume that $x \notin S$ since after the removal of U'' removing x does not disconnect anything from the graph. Similarly, we can also assume that $V_2' \nsubseteq S$. Then $H_t - S$ has at most 3 components. To obtain three components, the following must hold:

(i) $u'_{T+l} \in S$ or $v'_{T+l} \in S$ for all $l \in [T]$ (but $V_2' \nsubseteq S$), and

(ii) $V_1' \subseteq S$.

Hence $w(S) \geq T' + 2T = \lceil 2t \rceil + T$ and

$$\omega(H_t - S) = 3 \leq \frac{\lceil 2t \rceil + T}{t} = \frac{w(S)}{t}.$$

To obtain two components, either (i) or (ii) must hold; in both cases $w(S) \geq \lceil 2t \rceil$ and

$$\omega(H_t - S) = 2 \leq \frac{\lceil 2t \rceil}{t} \leq \frac{w(S)}{t}.$$

Case 3: $V_3' \subseteq S$.

First we show that the following assumptions can be made for S.

(1) $(U_1' \cup U_2' \cup U_3') \cap S = \emptyset$.

After the removal of V_3', removing any of the vertices of $U_1' \cup U_2' \cup U_3'$ does not disconnect anything from the graph.

(2) There exists at most one $l \in [T]$ for which $v'_{T+l} \notin S$, i.e. $|V_2' \setminus S| \leq 1$.

Suppose that there exist $l_1, l_2 \in [T]$ for which $l_1 \neq l_2$ and $v'_{T+l_1}, v'_{T+l_2} \notin S$. By assumption (1), considering the cutset $S' = S \cup \{v'_{T+l_2}\}$ instead of S increases both the number of components and the weight of the removed vertex set by 1. Hence it is enough to show that

$$\omega(H_t - S') \leq \frac{w(S')}{t}$$

since it implies

$$\omega(H_t - S) = \omega(H_t - S') - 1 \leq \frac{w(S')}{t} - 1 = \frac{w(S) + 1}{t} - 1 \leq \frac{w(S)}{t},$$

where the last inequality is valid since $t \geq 1$.

(3) For all $l \in [T]$ if $v_l' \in S$, then $v_l'' \notin S$.

After the removal of V_3' and v_l', removing v_l'' does not disconnect anything from the graph.

(4) For all $l \in [T]$ if $v_l' \notin S$, then $v_l'' \in S$.

Suppose that there exists $l \in [T]$ for which $v_l', v_l'' \notin S$. By assumption (1), considering the cutset $S' = S \cup \{v_l''\}$ instead of S increases both the number of components and the weight of the removed vertex set by 1. Hence, similarly as in assumption (2), it is enough to show that

$$\omega(H_t - S') \leq \frac{w(S')}{t}.$$

(5) $\left|(V_1' \cup V'') \cap S\right| = T$.

It follows directly from assumptions (3) and (4).

Case 3.1: $(V_3' \subseteq S$ and$)$ $U'' \subseteq S$.

Now we can assume that $x \notin S$ since after the removal of U'' removing x does not disconnect anything from the graph. Similarly, by assumption (2), we can also assume that $V_2' \nsubseteq S$, i.e. $|V_2' \cap S| = T - 1$. Hence

$$\begin{aligned} w(S) &= |V_3'| + |U''| + \left|(V_1' \cup V'') \cap S\right| + |V_2' \cap S| \\ &= (aT - 2T) + T' + T + (T - 1) = aT + T' - 1 \end{aligned}$$

and every component of $H_t - S$ contains exactly one of the vertices $u_1', \ldots, u_{bT-1}', x$, i.e.

$$\omega(H_t - S) = bT = \frac{aT}{t} \leq \frac{aT + T' - 1}{t} = \frac{w(S)}{t}.$$

Case 3.2: ($V_3' \subseteq S$ and) $U'' \cap S = \emptyset$.
In this case we can make some further assumptions for S.

(6) If $V_1' \subseteq S$, then $x \notin S$.

After the removal of V_1' removing x does not disconnect anything from the graph.

(7) If $V_1' \nsubseteq S$, then $x \in S$.

Suppose that $x \notin S$. Then considering the cutset $S' = S \cup \{x\}$ instead of S increases the number of components by 1 and the weight of the removed vertex set by t. Hence it is enough to show that $\omega(H_t - S') \leq w(S')/t$ since it implies

$$\omega(H_t - S) = \omega(H_t - S') - 1 \leq \frac{w(S')}{t} - 1 = \frac{w(S) + t}{t} - 1 = \frac{w(S)}{t}.$$

(8) $V_2' \subseteq S$.

Suppose that $V_2' \nsubseteq S$. Then by assumption (2), there exists $l \in [T]$ for which $V_2' \setminus S = \{v_{T+l}'\}$. But by assumption (1), considering the cutset $S' = S \cup \{v_{T+l}'\}$ instead of S increases both the number of components and the weight of the removed vertex set by 1. Then, similarly as in assumption (2), it is enough to show that $\omega(H_t - S') \leq w(S')/t$.

Case 3.2.1: ($V_3' \subseteq S$, $U'' \cap S = \emptyset$ and) $V_1' \subseteq S$.

Hence

$$w(S) = |V_3'| + |V_2'| + |V_1'| = aT$$

and

$$\omega(H_t - S) = bT = \frac{w(S)}{t}.$$

Case 3.2.2: ($V_3' \subseteq S$, $U'' \cap S = \emptyset$ and) $V_1' \nsubseteq S$.

Hence

$$w(S) = |V_3'| + |V_2'| + |(V_1' \cup V'') \cap S| + w(x) = aT + t$$

and

$$\omega(H_t - S) = bT + 1 = \frac{w(S)}{t}.$$

Therefore H_t is weighted t-tough with respect to w (meaning that the weighted toughness of H_t is at least t).

Consider the cutset

$$S = V_1' \cup V_2' \cup V_3'.$$

Since $w(S) = aT$ and

$$\omega(H_t - S) = bT = \frac{w(S)}{t},$$

the weighted toughness of H_t with respect to w is at most t.

Thus the weighted toughness of H_t with respect to w is exactly t. □

Deleting an edge may decrease the weighted toughness, and now we delete edges not induced by U'' until the weighted toughness with respect to the weight function w remains at least t but the deletion of any other edge not induced by U'' would result in a graph with weighted toughness less than t. Let H_t' denote the obtained graph.

According to the following claim we could not delete the edges induced by V_1' or incident to any of the vertices of $\{x\} \cup V_2' \cup U''$.

Claim 4.15. Let $t \geq 1$ be a rational number. For any edge $e \in E(H_t)$ induced by V_1' or incident to any of the vertices of $\{x\} \cup V_2' \cup U''$, there exists a cutset $S = S(e) \subseteq V(H_t)$ in $H_t - e$ for which

$$\omega\big((H_t - e) - S\big) > \frac{w(S)}{t}.$$

Proof. Let $e \in E(H_t)$ be an arbitrary edge induced by V_1' or incident to any of the vertices of $\{x\} \cup V_2' \cup U''$.

Case 1: e is incident to a vertex of $\{x\} \cup V_2'$.

Let $y \in \{x\} \cup V_2'$ denote one of the endpoints of e, and let z denote the other one. Let S be the neighborhood of the vertex y except for z. Since y has degree $\lceil 2t \rceil$ and all of its neighbors have weight 1,

$$w(S) = \lceil 2t \rceil - 1.$$

Since the removal of S from $H_t - e$ leaves the vertex y isolated,

$$\omega\big((H_t - e) - S\big) \geq 2 = \frac{2t}{t} > \frac{\lceil 2t \rceil - 1}{t} = \frac{w(S)}{t}.$$

Case 2: e is incident to a vertex of U''.

If e is incident to a vertex of U'', then either it is incident to a vertex of $\{x\} \cup V_2'$ and this case was already settled in Case 1, or it is induced by U'' and therefore it was not allowed to be deleted.

Case 3: e is induced by V_1', i.e. $e = v_{l_1}' v_{l_2}'$ for some $l_1, l_2 \in [T]$, $l_1 \neq l_2$.

Then

$$S = \big(V_1' \setminus \{v_{l_1}', v_{l_2}'\}\big) \cup \{v_{l_1}'', v_{l_2}''\} \cup V_2' \cup V_3' \cup \{x\}$$

is a cutset in $H_t - e$ such that

$$w(S) = (T-2) + 2 + T + (aT - 2T) + t = aT + t$$

and

$$\omega\big((H_t - e) - S\big) = bT + 2 = \frac{aT + t}{t} + 1 = \frac{w(S)}{t} + 1 > \frac{w(S)}{t}.$$

□

Claim 4.16. Let $t \geq 1$ be a rational number and $H_t'' = H_t' - \{x\}$. Then the following hold.

(i) The graph H_t'' is connected.

(ii) For any cutset S of H_t'',

$$\omega(H_t'' - S) \leq \frac{|S|}{t} + 1.$$

(iii) If $V_1' \subseteq S$ holds for a cutset S of H_t'', then

$$\omega(H_t'' - S) \leq \frac{|S|}{t}.$$

(iv) For any edge $e \in E(H_t'')$ not induced by U'' there exists a vertex set $S = S(e)$ whose removal from $H_t'' - e$ disconnects the graph and

$$\omega\big((H_t'' - e) - S\big) > \frac{|S|}{t}.$$

Proof.

(i) Suppose to the contrary that H_t'' is not connected. Then x is a cut-vertex in H_t'. Since the weighted toughness of H_t' with respect to w is t,

$$2 \leq \omega\big(H_t' - \{x\}\big) \leq \frac{w(x)}{t} = \frac{t}{t} = 1,$$

which is a contradiction.

(ii) Let S be an arbitrary cutset of H_t''. Since S is a cutset in H_t'', the vertex set $S \cup \{x\}$ is a cutset in H_t', and

$$\omega(H_t'' - S) = \omega\big(H_t' - (S \cup \{x\})\big) \leq \frac{w(S \cup \{x\})}{t} = \frac{|S| + t}{t} = \frac{|S|}{t} + 1.$$

(iii) Let S be a cutset of H_t'' for which $V_1' \subseteq S$. We can assume that $U'' \cap S = \emptyset$ since removing any of the vertices of U'' from H_t'' does not disconnect anything from the graph. Then all the neighbors of the vertex x belong to the same component in $H_t'' - S$, so S is a cutset in H_t' as well and

$$\omega(H_t'' - S) = \omega(H_t' - S) \leq \frac{w(S)}{t} = \frac{|S|}{t},$$

where the last equality is valid since $x \notin S$.

(iv) Let $e \in E(H''_t)$ be an arbitrary edge not induced by U''. Then by the properties of H'_t, there exists a vertex set $S \subseteq V(H'_t)$ whose removal from $H'_t - e$ disconnects the graph and

$$\omega\big((H'_t - e) - S\big) > \frac{w(S)}{t} \geq \frac{|S|}{t},$$

where the last inequality is valid since $t \geq 1$. Let $S' = S \setminus \{x\}$. Then

$$\omega\big((H''_t - e) - S'\big) \geq \omega\big((H'_t - e) - S\big) > \frac{|S|}{t} \geq \frac{|S'|}{t}.$$

□

4.4.3 The cutsets X and $Y_1, \ldots, Y_T$ in H''_t when $t \geq 1$

Let

$$X = V'_1 \cup V'_2 \cup V'_3$$

and for all $l \in [T]$ let

$$Y_l = \big(V'_1 \setminus \{v'_l\}\big) \cup \{v''_l\} \cup V'_2 \cup V'_3.$$

Proposition 4.17. The sets X and $Y_1, \ldots, Y_T$ are all cutsets in H''_t and

$$\omega(H''_t - X) = \frac{|X|}{t},$$

and

$$\omega(H''_t - Y_l) = \frac{|Y_l|}{t} + 1$$

for all $l \in [T]$.

Proof. It is easy to see that

$$\omega(H''_t - X) = bT = \frac{aT}{t} = \frac{|X|}{t}$$

and

$$\omega(H''_t - Y_l) = bT + 1 = \frac{aT}{t} + 1 = \frac{|Y_l|}{t} + 1.$$

□

4.4.4 The proof of Theorem 4.1 when $t \geq 1$

Theorem 4.18 (Katona, Kovács, Varga, [2])**.** For any rational number $t \geq 1$, the problem Min-t-Tough is DP-complete.

Proof. Let $t \geq 1$ be a rational number. In Proposition 4.5 we already proved that the problem Min-t-Tough is in DP. To show that it is DP-hard, we reduce the variant of α-Critical defined in Proposition 2.21 to it.

Let $T = \lceil t \rceil$, and $T' = \lceil 2t \rceil - \lceil t \rceil$, and $M = \lceil 2\lceil t \rceil / \lceil 2t \rceil \rceil$ as before. Let a, b be the smallest positive integers such that $b \geq 3$ and $t = a/b$, let G be an arbitrary 3-connected graph on the vertices $v_1, \dots, v_n$ with $n \geq t+1$, let k be a positive integer that is divisible by a and let $G_{t,k}$ be defined as follows. For all $i \in [n], j \in [k], m \in [M]$ let

$$V_{i,j,m} = \left\{ v_{i,j,l,m} \;\middle|\; l \in [T] \right\}.$$

For all $i \in [n]$ let

$$V_i = \bigcup_{\substack{j \in [k], \\ m \in [M]}} V_{i,j,m}$$

and place a clique on the vertices of V_i. For all $i_1, i_2 \in [n]$ if $v_{i_1} v_{i_2} \in E(G)$, then place a complete bipartite graph on $(V_{i_1}; V_{i_2})$. (This subgraph is denoted by $\tilde{G}$ in Figure 4.6.) For all $i \in [n], j \in [k], m \in [M]$ "glue" the graph H''_t to the vertex set $V_{i,j,m}$ by identifying $v_{i,j,l,m}$ with the vertex v'_l of H''_t for all $l \in [T]$. For all $i \in [n], j \in [k], m \in [M]$ let $H^{i,j,m}$, $U''_{i,j,m}$ and $X_{i,j,m}$ denote the (i,j,m)-th copies of H''_t, U'' and X, respectively. For all $i \in [n], j \in [k], l \in [T'], m \in [M]$ let $u''_{i,j,l,m}$ denote the (i,j,m)-th copy of u''_l, and for all $i \in [n], j \in [k], l \in [T], m \in [M]$ let $Y_{i,j,l,m}$ denote the (i,j,m)-th copy of Y_l. For all $j \in [k], m \in [M]$ add the vertex set

$$W_{j,m} = \left\{ w_{j,l,m} \;\middle|\; l \in [T'] \right\}$$

to the graph and for all $i \in [n], j \in [k], l \in [T'], m \in [M]$ connect $w_{j,l,m}$ to $u''_{i,j,l,m}$. Let

$$V = \bigcup_{i \in [n]} V_i, \qquad U'' = \bigcup_{\substack{i \in [n], \\ j \in [k], \\ m \in [M]}} U''_{i,j,m}, \qquad W = \bigcup_{\substack{j \in [k], \\ m \in [M]}} W_{j,m},$$

and let

$$U = \left(\bigcup_{\substack{i \in [n],\\ j \in [k],\\ m \in [M]}} V(H^{i,j,m}) \right) \setminus V.$$

Add the vertex set

$$W' = \{w'_1, \ldots, w'_{(MT'/t-1)k}\}$$

to the graph and place the bipartite graph $H^{**}_{t,k}$ on $(W; W')$. See Figure 4.6. Now k is part of the input of the problem α-CRITICAL, therefore the graph $H^{**}_{t,k}$ must be constructed in polynomial time and by Theorem 2.11, this can be done. On the other hand, t is not part of the input of the problem MIN-t-TOUGH, therefore the graph H''_t can be constructed in advance. Hence, $G_{t,k}$ can be constructed from G in polynomial time.

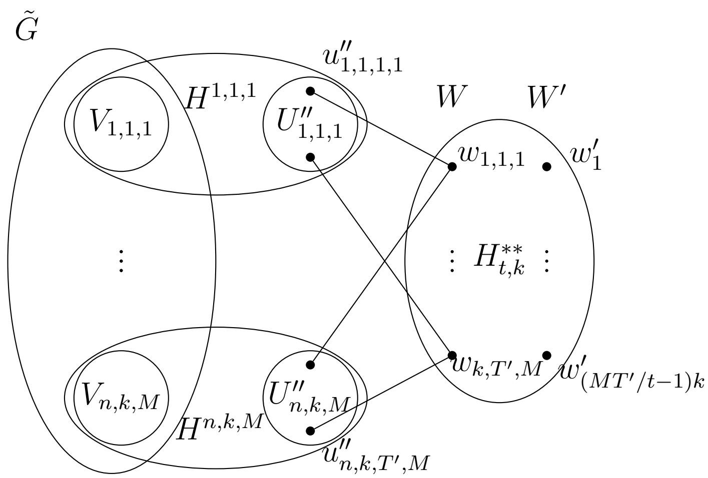

Figure 4.6: The graph $G_{t,k}$, when $t \geq 1$.

To show that G is α-critical with $\alpha(G) = k$ if and only if $G_{t,k}$ is minimally t-tough, first we prove the following lemma.

Lemma 4.19. Let G be an arbitrary 3-connected graph on $n \geq t+1$ vertices with $\alpha(G) \leq k$. Then $G_{t,k}$ is t-tough.

Proof. Let $S \subseteq V(G_{t,k})$ be a cutset in $G_{t,k}$. We need to show that $\omega(G_{t,k} - S) \leq |S|/t$.

Case 1: $W \subseteq S$.

After the removal of W, the vertices of W' are isolated, therefore we can assume that $W' \cap S = \emptyset$.

Let

$$C = C(S) = \big\{(i,j,m) \in [n] \times [k] \times [M] \;\big|\; V_{i,j,m} \subseteq S\big\},$$
$$c_{i,j,m} = \big|V(H^{i,j,m}) \cap S\big| - T$$

for all $(i,j,m) \in C$, and

$$d_{i,j,m} = \big|V(H^{i,j,m}) \cap S\big|$$

for all $(i,j,m) \in \big([n] \times [k] \times [M]\big) \setminus C$. Let

$$D = D(S) = \left\{(i,j,m) \in \big([n] \times [k] \times [M]\big) \setminus C \;\middle|\; d_{i,j,m} > 0\right\}.$$

Using these notations it is clear that

$$|S| = |C| \cdot T + \sum_{(i,j,m) \in C} c_{i,j,m} + \sum_{(i,j,m) \in D} d_{i,j,m} + kMT'.$$

By the assumption that $W \subseteq S$, in $G_{t,k} - S$ the $(MT'/t - 1)k$ vertices of W' are isolated. Since $\alpha\big(G_{t,k}[V]\big) = \alpha(G)$, the removal of $V \cap S$ from $G_{t,k}[V]$ leaves at most $\alpha(G)$ components. By Claim 4.16, for any $(i,j,m) \in C$ the removal of $V(H^{i,j,m}) \cap S$ from $H^{i,j,m}$ leaves at most

$$\max\left(\frac{|V(H^{i,j,m}) \cap S|}{t}, 1\right) = \max\left(\frac{c_{i,j,m} + T}{t}, 1\right) = \frac{c_{i,j,m} + T}{t}$$

components. By Claim 4.16, for any $(i,j,m) \in D$ the removal of $V(H^{i,j,m}) \cap S$ from $H^{i,j,m}$ leaves at most

$$\max\left(\frac{|V(H^{i,j,m}) \cap S|}{t} + 1, 1\right) = \max\left(\frac{d_{i,j,m}}{t} + 1, 1\right) = \frac{d_{i,j,m}}{t} + 1$$

components, but the component of the remaining vertices of $V_{i,j,m}$ has been already counted. Hence

$$\begin{aligned}
\omega(G_{t,k} - S) &\leq \left(\frac{MT'}{t} - 1\right)k + \alpha(G) + \sum_{(i,j,m) \in C} \frac{c_{i,j,m} + T}{t} + \sum_{(i,j,m) \in D} \frac{d_{i,j,m}}{t} \\
&\leq \frac{kMT' + |C| \cdot T + \sum_{(i,j,m) \in C} c_{i,j,m} + \sum_{(i,j,m) \in D} d_{i,j,m}}{t} = \frac{|S|}{t}.
\end{aligned}$$

Case 2: $W \nsubseteq S$.

There are four types of components in $G_{t,k} - S$:

(a) components containing at least one vertex of V,

(b) components containing at least one vertex of U but no vertices of V,

(c) components containing at least one vertex of W but no vertices of $U \cup V$,

(d) isolated vertices of W'.

Let $w_{j_0,l_0,m_0} \in W \setminus S$ be fixed. First we show that the following assumptions can be made for S.

(1) $S \cap U'' = \emptyset$.

Obviously, the number of vertices of W that belong to a component of type (c) is at most $|S \cap U''|/n$. Since the neighborhood of any vertex of U'' spans a clique in $G_{t,k}$, considering the cutset

$$S' = (S \setminus U'') \cup \{w \in W \mid w \text{ belongs to a component of type (c)}\}$$

instead of S can only increase the number of components of types (a), (b) and (d), while it decreases the number of components of type (c) to 0, i.e., by at most $|S \cap U''|/n$. Hence,

$$|S'| \le |S| - |S \cap U''| + \frac{|S \cap U''|}{n}$$

and

$$\omega(G_{t,k} - S') \ge \omega(G_{t,k} - S) - \frac{|S \cap U''|}{n}.$$

Then it is enough to prove that $\omega(G_{t,k} - S') \le |S'|/t$ since it implies

$$\begin{aligned}
\omega(G_{t,k} - S) &\le \omega(G_{t,k} - S') + \frac{|S \cap U''|}{n} \le \frac{|S'|}{t} + \frac{|S \cap U''|}{n} \\
&\le \frac{|S| - |S \cap U''| + |S \cap U''|/n}{t} + \frac{|S \cap U''|}{n} \\
&= \frac{|S|}{t} - |S \cap U''| \cdot \frac{n-t-1}{nt} \le \frac{|S|}{t},
\end{aligned}$$

where the last inequality is valid since $n \ge t + 1$.

(2) There are no components of type (c) in $G_{t,k} - S$.

It follows directly from assumption (1).

(3) $\big|\{i \in [n] \mid V_{i,j_0,m_0} \subseteq S\}\big| \leq \lceil T'/t \rceil$.

Let

$$I = I(w_{j_0,l_0,m_0}, S) = \big\{i \in [n] \,\big|\, V_{i,j_0,m_0} \subseteq S\big\}$$

and suppose that $|I| \geq \lceil T'/t \rceil + 1$. By assumption (1), the component of w_{j_0,l_0,m_0} contains every vertex of $\bigcup_{i=1}^{n} U_{i,j_0,m_0}$ and therefore all the remaining vertices of W_{j_0,m_0}. Now considering the cutset

$$S' = S \cup \{W_{j_0,m_0}\}$$

instead of S increases the number of removed vertices by at most T', and it increases the number of components by at least $\lceil T'/t \rceil$ since it disconnects the vertex sets U_{i,j_0,m_0}, $i \in I$ from each other. Then it is enough to show that $\omega(G_{t,k} - S') \leq |S'|/t$ since it implies

$$\omega(G_{t,k} - S) \leq \omega(G_{t,k} - S') - \left\lceil \frac{T'}{t} \right\rceil \leq \frac{|S'|}{t} - \left\lceil \frac{T'}{t} \right\rceil \leq \frac{|S| + T'}{t} - \left\lceil \frac{T'}{t} \right\rceil \leq \frac{|S|}{t}.$$

Proceeding further, we can obtain a cutset S^* for which $W \subseteq S^*$ holds; and such sets were already handled in Case 1.

(4) $(G_{t,k} - S)[V]$ is connected, i.e. there is only one component of type (a).

Since $t \geq 1$,

$$\left\lceil \frac{T'}{t} \right\rceil \leq \left\lceil \frac{t+1}{t} \right\rceil = 1 + \left\lceil \frac{1}{t} \right\rceil \leq 2.$$

Since G is 3-connected, assumption (2) implies that $(G_{t,k} - S)[V]$ is connected.

Using the previous notations,

$$|S| = |C| \cdot T + \sum_{(i,j,m) \in C} c_{i,j,m} + \sum_{(i,j,m) \in D} d_{i,j,m} + |S \cap (W \cup W')|.$$

By assumption (2), there are no components of type (c), and by assumption (4), there is only one component of type (a). By the properties of $H^{**}_{t,k}$, the removal of $S \cap (W \cup W')$ from $H^*_{t,k}$ leaves at most

$$\max\left(\frac{|S \cap (W \cup W')|}{t}, 1\right)$$

components, one of them is the component of w_{j_0,l_0,m_0}, hence there are at most

$$\max\left(\frac{|S \cap (W \cup W')|}{t}, 1\right) - 1 \leq \frac{|S \cap (W \cup W')|}{t}$$

components of type (d). Similarly as before, for any $(i, j, m) \in C$ the removal of $V(H^{i,j,m}) \cap S$ from $H^{i,j,m}$ leaves at most

$$\frac{c_{i,j,m} + T}{t}$$

components, all of which can be of type (b). For any $(i, j, m) \in D$ the removal of $V(H^{i,j,m}) \cap S$ from $H^{i,j,m}$ leaves at most

$$\frac{d_{i,j,m}}{t} + 1$$

components; the component of the remaining vertices of $V_{i,j,m}$ is of type (a), all the others can be of type (b). By assumption (1), all the vertices of $\bigcup_{i=1}^{n} U_{i,j_0,m_0}$ belong to the component of w_{j_0,l_0,m_0}, hence the component of w_{j_0,l_0,m_0} has been counted multiple times (more than once). Therefore,

$$\omega(G_{t,k} - S) \leq \left(1 + \sum_{(i,j,m) \in C} \frac{c_{i,j,m} + T}{t} + \sum_{(i,j,m) \in D} \frac{d_{i,j,m}}{t} + \frac{|S \cap (W \cup W')|}{t}\right) - 1 = \frac{|S|}{t}.$$

Thus, $\tau(G_{t,k}) \geq t$. □

Now we return to the proof of Theorem 4.18 and we show that G is α-critical with $\alpha(G) = k$ if and only if $G_{t,k}$ is minimally t-tough.

Let us assume that G is α-critical with $\alpha(G) = k$. By Lemma 4.19, the graph $G_{t,k}$ is t-tough, i.e. $\tau(G_{t,k}) \geq t$.

Let I be an independent vertex set of size $\alpha(G)$ in G, and recall the definition of the sets X and $Y_1, \dots, Y_T$ constructed in Subsection 4.4.3. Let

$$J = \left\{ i \in [n] \;\middle|\; v_i \in I \right\}$$

and

$$S = \left(\bigcup_{i \in J} Y_{i,1,1,1} \right) \cup \left(\bigcup_{\substack{i \in J,\\ j \in [k] \setminus \{1\},\\ m \in [M] \setminus \{1\}}} X_{i,j,m} \right) \cup \left(\bigcup_{\substack{i \notin J,\\ j \in [k],\\ m \in [M]}} X_{i,j,m} \right) \cup W.$$

Then S is a cutset in $G_{t,k}$ with

$$|S| = nkMaT + kMT'$$

and after the removal of S from $G_{t,k}$, the vertices of W' are isolated and the other components of $G_{t,k} - S$ are exactly the components of $H^{i,j,m} - \big(S \cap V(H^{i,j,m})\big)$ for all $(i, j, m) \in [n] \times [k] \times [M]$. By Proposition 4.17,

$$\omega(H^{i,j,m} - X_{i,j,m}) = bT$$

and

$$\omega(H^{i,j,m} - Y_{i,1,1,1}) = bT + 1$$

for all $(i, j, m) \in [n] \times [k] \times [M]$. Since $|J| = \alpha(G)$ and

$$|W'| = \left(\frac{MT'}{t} - 1 \right) k,$$

it follows that

$$\begin{aligned} \omega(G_{t,k} - S) = nkMbT + \alpha(G) + \left(\frac{MT'}{t} - 1 \right) k &= nkMbT + \frac{kMT'}{t} \\ &= \frac{nkMaT + kMT'}{t} = \frac{|S|}{t}, \end{aligned}$$

so $\tau(G_{t,k}) \le t$.

Therefore, $\tau(G_{t,k}) = t$.

Let $e \in E(G_{t,k})$ be an arbitrary edge. We need to show that $\tau(G_{t,k}-e) < t$. Now we have four cases.

Case 1: e has an endpoint in U''.

Then this endpoint has degree $\lceil 2t \rceil - 1$ in $G_{t,k} - e$, so

$$\tau(G_{t,k} - e) \leq \frac{\lceil 2t \rceil - 1}{2} < \frac{2t}{2} = t.$$

Case 2: e has an endpoint in W'.

By the properties of $H^*_{t,k}$, there exists a cutset $S \subseteq W$ in $H^*_{t,k} - e$ for which

$$\omega\big((H^*_{t,k} - e) - S\big) > \frac{|S|}{t}.$$

Note that S is also a cutset in $G_{t,k} - e$ and

$$\omega\big((G_{t,k} - e) - S\big) = \omega\big((H^*_{t,k} - e) - S\big) > \frac{|S|}{t},$$

so $\tau(G_{t,k} - e) < t$.

Case 3: e is induced by H^{i_0,j_0,m_0} for some $i_0 \in [n], j_0 \in [k], m_0 \in [M]$.

The case when e is induced by U''_{i_0,j_0,m_0} was already covered in Case 1. So assume that e is not induced by U''_{i_0,j_0,m_0}. Then by Claim 4.16, there exists a vertex set $S \subseteq V(H''_t)$ for which

$$\omega\big((H''_t - e) - S\big) > \frac{|S|}{t}.$$

Consider the (i_0, j_0, m_0)-th copy of the vertex set S in $G_{t,k} - e$; let us denote it with S_{i_0,j_0,m_0}. If $V_{i_0,j_0,m_0} \subseteq S_{i_0,j_0,m_0}$, then S_{i_0,j_0,m_0} is a cutset in $G_{t,k} - e$ and

$$\omega\big((G_{t,k} - e) - S_{i_0,j_0,m_0}\big) = \omega\big((H''_t - e) - S\big) > \frac{|S|}{t},$$

so $\tau(G_{t,k} - e) < t$. Assume that $V_{i_0,j_0,m_0} \nsubseteq S_{i_0,j_0,m_0}$. Let I be an independent vertex set of size $\alpha(G)$ in G that contains v_{i_0} (by Proposition 2.18, such an independent vertex set exists). Let

$$J = \big\{ i \in [n] \;\big|\; v_i \in I \big\}$$

and

$$S' = S_{i_0,j_0,m_0} \cup \left(\bigcup_{\substack{j\in[k],\\ m\in[M],\\ (j,m)\neq(j_0,m_0)}} X_{i_0,j,m} \right) \cup \left(\bigcup_{i\in J\setminus\{i_0\}} Y_{i,1,1,1} \right)$$
$$\cup \left(\bigcup_{\substack{i\in J\setminus\{i_0\},\\ j\in[k]\setminus\{1\},\\ m\in[M]\setminus\{1\}}} X_{i,j,m} \right) \cup \left(\bigcup_{\substack{i\notin J,\\ j\in[k],\\ m\in[M]}} X_{i,j,m} \right) \cup W.$$

Then S' is a cutset in $G_{t,k} - e$ with

$$|S'| = |S| + (nkM-1)aT + kMT'$$

and similarly as before,

$$\begin{aligned} \omega\big((G_{t,k}-e)-S'\big) &> \frac{|S|}{t} + (nkM-1)bT + \alpha(G) + \left(\frac{MT'}{t} - 1\right)k \\ &= \frac{|S|}{t} + (nkM-1)bT + \frac{kMT'}{t} = \frac{|S'|}{t}, \end{aligned}$$

so $\tau(G_{t,k}-e) < t$.

Case 4: e connects two vertices of V.

Since the case when e is induced by H^{i_0,j_0,m_0} for some $i_0 \in [n], j_0 \in [k], m_0 \in [M]$ was settled in Case 3, we can assume that there do not exist $i \in [n], j \in [k], m \in [M]$ for which e is induced by $V_{i,j,m}$. By Lemma 2.19, the graph $G_{t,k}[V]$ is α-critical, so in $G_{t,k}[V] - e$ there exists an independent vertex set I of size $\alpha(G)+1$. Let

$$J = \big\{(i,j,l,m) \in [n]\times[k]\times[T]\times[M] \bigm| v_{i,j,l,m} \in I\big\},$$

$$J_1' = \big\{(i,j,m) \in [n]\times[k]\times[M] \bigm| \exists! l \in [T]:\ v_{i,j,l,m} \in I\big\},$$

and

$$J_2' = \big\{(i,j,m) \in [n]\times[k]\times[M] \bigm| \nexists l \in [T]:\ v_{i,j,l,m} \in I\big\}.$$

By the assumption that there do not exist $i \in [n], j \in [k], m \in [M]$ for which e is induced by $V_{i,j,m}$,

$$J_1' \cup J_2' = [n] \times [k] \times [M],$$

so

$$S = \left(\bigcup_{(i,j,l,m) \in J} Y_{i,j,l,m} \right) \cup \left(\bigcup_{(i,j,m) \in J_2'} X_{i,j,m} \right) \cup W$$

is a (well-defined) cutset in $G_{t,k} - e$. Then

$$|S| = nkMaT + kMT'$$

and similarly as before,

$$\begin{aligned} \omega\big((G_{t,k} - e) - S\big) &= nkMbT + \alpha(G) + 1 + \left(\frac{MT'}{t} - 1\right) k \\ &= \frac{nkMaT}{t} + \frac{kMT'}{t} + 1 > \frac{|S|}{t}, \end{aligned}$$

so $\tau(G_{t,k} - e) < t$.

Now let us assume that G is not α-critical with $\alpha(G) = k$, i.e. either $\alpha(G) \neq k$ or even though $\alpha(G) = k$, the graph G is not α-critical.

Case I: $\alpha(G) > k$.
Let I be an independent vertex set of size $\alpha(G)$ in G. Let

$$J = \left\{ i \in [n] \;\middle|\; v_i \in I \right\}$$

and

$$S = \left(\bigcup_{i \in J} Y_{i,1,1,1} \right) \cup \left(\bigcup_{\substack{i \in J, \\ j \in [k] \setminus \{1\}, \\ m \in [M] \setminus \{1\}}} X_{i,j,m} \right) \cup \left(\bigcup_{\substack{i \notin J, \\ j \in [k], \\ m \in [M]}} X_{i,j,m} \right) \cup W.$$

Then S is a cutset in $G_{t,k} - e$ with

$$|S| = nkMaT + kMT'$$

and similarly as before,

$$\omega\big((G_{t,k}-e)-S\big) = nkMbT + \alpha(G) + \left(\frac{MT'}{t} - 1\right)k > nkMbT + \frac{kMT'}{t}$$
$$= \frac{nkMaT + kMT'}{t} = \frac{|S|}{t},$$

so $\tau(G_{t,k}) < t$, which means that $G_{t,k}$ is not minimally t-tough.

Case II: $\alpha(G) \leq k$.

Since G is not α-critical with $\alpha(G) = k$ there exists an edge $e \in E(G)$ such that $\alpha(G-e) \leq k$. By Lemma 4.19, the graph $(G-e)_{t,k}$ is t-tough, but it can be obtained from $G_{t,k}$ by edge-deletion, which means that $G_{t,k}$ is not minimally t-tough. $\square$

Therefore the problem Min-1-Tough is DP-complete, so by Claim 2.6, we can conclude the following.

Corollary 4.20 (Katona, Kovács, Varga, [2])**.** Recognizing almost minimally 1-tough graphs is DP-complete.

Let Almost-Min-1-Tough denote the problem of determining whether a given graph is almost minimally 1-tough.

4.5 Minimally t-tough graphs with $t \leq 1/2$

The case when $t \leq 1/2$ is special in some sense: graphs with toughness at most $1/2$ can have cut-vertices. Unlike in the previous cases, we reduce Almost-Min-1-Tough to this problem. But first, again, we construct an auxiliary graph.

4.5.1 The auxiliary graph H_t' when $t \leq 1/2$

Let $t \leq 1/2$ be a positive rational number. Let a, b be relatively prime positive integers such that $t = a/b$ and let H_t be constructed as follows. Let

$$V = \{v_1, v_2, \dots, v_a\}, \qquad U = \{u_1, u_2, \dots, u_{b-a}\}, \qquad W = \{w_1, w_2, \dots, w_a\}.$$

Place a clique on the vertices of V, connect every vertex of V to every vertex of U, and connect v_i to w_i for all $i \in [n]$. See Figure 4.7.

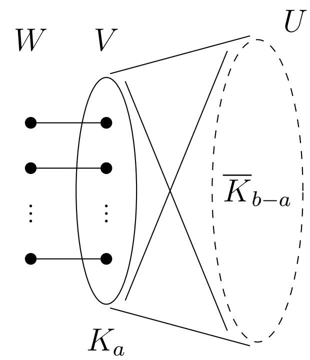

Figure 4.7: The graph H_t when $t \leq 1/2$.

Proposition 4.21. Let $t \leq 1/2$ be a positive rational number. Then $\tau(H_t) = t$.

Proof. Let S be an arbitrary cutset of H_t. We can assume that $S \cap (U \cup W) = \emptyset$ since removing any of the vertices of $U \cup W$ does not disconnect anything in the graph. Then $S \subseteq V$, so

$$\omega(H_t - S) = \begin{cases} a + (b-a) = b & \text{if } S = V, \\ |S| + 1 & \text{if } S \neq V, \end{cases}$$

which implies that

$$\tau(H_t) = \min \left\{ \frac{|S|}{\omega(H_t - S)} \;\middle|\; S \subseteq V, S \neq \emptyset \right\} = \frac{a}{b} = t.$$

□

By repeatedly deleting some edges of H_t, eventually we obtain a minimally t-tough graph; let us denote it with H'_t (i.e. if there exists an edge whose removal does not decrease the toughness, then we delete it). Obviously, we could not delete the edges between V and W, so the vertices of W still have degree 1 in H'_t.

Note that V is a tough set of H'_t. For further reference (to avoid confusion with other sets denoted by V), we introduce a new name for it.

Notation 4.22. Let S_t denote the tough set V in H'_t.

4.5.2 "Gluing"

Definition 4.23. Let H be a graph with a vertex u of degree 1, and let v be the neighbor of u. Let G be an arbitrary graph, and "glue" $H - \{u\}$ separately to all vertices of G by identifying each vertex of G with v. Let $G \oplus_v H$ denote the obtained graph. (See Figure 4.8.)

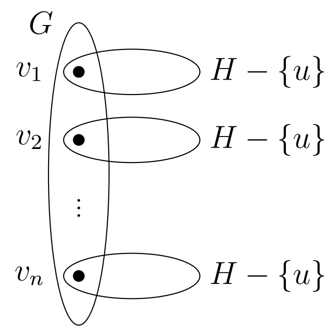

Figure 4.8: The graph $G \oplus_v H$.

4.5.3 The proof of Theorem 4.1 when $t \leq 1/2$

Theorem 4.24 (Katona, Kovács, Varga, [2])**.** For any positive rational number $t \leq 1/2$, the problem Min-t-Tough is DP-complete.

Proof. Let $t \leq 1/2$ be a positive rational number. In Proposition 4.5 we already proved that the problem Min-t-Tough is in DP. To show that it is DP-hard, we reduce Almost-Min-1-Tough to it.

Let G be an arbitrary graph and $n = |V(G)|$. Consider the graph H'_t and let $u \in U$ be an arbitrary vertex of H'_t having degree 1, and let v be its neighbor. Let

$$H''_t = H'_t - \{u\}$$

and let H^i denote the i-th copy of H''_t "glued" to the vertex $v_i \in V(G)$ for all $i \in [n]$. (For examples see Figures A.4 and A.5.)

Now we show that G is almost minimally 1-tough if and only if $G_t = G \oplus_v H'_t$ is minimally t-tough.

First, let G be almost minimally 1-tough. We need to show that G_t is minimally t-tough.

Let $S \subseteq V(G_t)$ be an arbitrary cutset of G_t. Let

$$C = C(S) = \bigl\{ i \in [n] \bigm| v_i \in V(G) \cap S \bigr\},$$

$$c_i = \bigl|V(H^i) \cap S\bigr| - 1$$

for all $i \in C$, and

$$d_i = \bigl|V(H^i) \cap S\bigr|$$

for all $i \in [n] \setminus C$ (see Figure 4.9). Finally, let

$$D = D(S) = \bigl\{ i \in [n] \setminus C \bigm| d_i > 0 \bigr\}.$$

Using these notations it is clear that

$$|S| = |C| + \sum_{i \in C} c_i + \sum_{i \in D} d_i.$$

By Proposition 2.7, the removal of $V(G) \cap S$ from G leaves at most $|V(G) \cap S| = |C|$ components. By Proposition 2.2, the removal of $V(H'_t) \cap S$ from H'_t leaves at most $|V(H'_t) \cap S|/t$ components. But for all $i \in [n] \setminus C$ we have already counted the component of $G'_t - S$ which contains v_i, and for all $i \in C$ we do not need to count the component $\{u\}$ of H'_t. Hence

$$\begin{aligned}\omega(G_t - S) &\le |C| + \sum_{i \in C} \left(\frac{c_i + 1}{t} - 1 \right) + \sum_{i \in D} \left(\frac{d_i}{t} - 1 \right) \\ &= \frac{|C| + \sum_{i \in C} c_i + \sum_{i \in D} d_i}{t} - |D| \le \frac{|S|}{t},\end{aligned}$$

which means that $\tau(G_t) \ge t$.

Now let S_0 be a tough set of H'_t. Since u has degree 1, we can assume that $u \notin S_0$. Let $S_0^1 \subseteq V(H^1)$ be the first copy of S_0. Obviously, S_0^1 is a cutset in G_t, and

$$\omega(G_t - S_0^1) = \omega(H'_t - S_0) = \frac{|S_0|}{t} = \frac{|S_0^1|}{t},$$

which means that $\tau(G_t) \le t$.

Therefore, $\tau(G_t) = t$.

Let $e \in E(G_t)$ be an arbitrary edge. We need to show that $\tau(G_t - e) < t$ for all $e \in E(G_t)$. Now we have two cases.

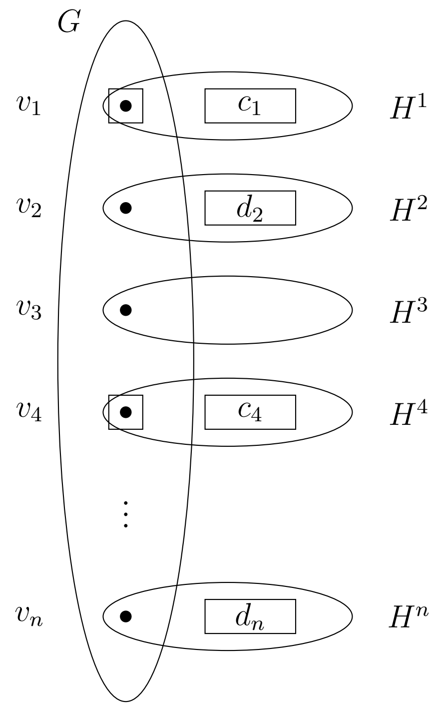

Figure 4.9: The graph G_t and the cutset S, when $t \leq 1/2$.

Case 1: $e \in E(G)$.

If e is a bridge in G, then $\tau(G_t - e) = 0 < t$. So assume that e is not a bridge in G. Let $S = S(e) \neq \emptyset$ be a vertex set in G guaranteed by Claim 2.6, and for all $i \in [n]$ let $S_t^i \subseteq V(H^i)$ be the i-th copy of the tough set S_t defined in Notation 4.22. (Note that $v \in S_t$ and $u \notin S_t$.) Let

$$J = J(S) = \left\{ i \in [n] \;\middle|\; v_i \in S \right\}$$

and consider the vertex set

$$S' = S \cup \left(\bigcup_{i \in J} S_t^i \right) = \bigcup_{i \in J} S_t^i.$$

Then S' is a cutset in $G_t - e$ with

$$|S'| = \sum_{i \in J} |S_t^i| = |S| \cdot |S_t|$$

and

$$\omega\big((G_t - e) - S'\big) > |S| + |S|\left(\frac{|S_t|}{t} - 1\right) = \frac{|S| \cdot |S_t|}{t} = \frac{|S'|}{t},$$

which means that $\tau(G_t - e) < t$.

Case 2: $e \in E(H^{i_0})$ for some $i_0 \in [n]$.

If e is a bridge in H_t', then $\tau(G_t - e) = 0 < t$. So assume that e is not a bridge in H_t' and let $S = S(e) \neq \emptyset$ be a vertex set in H_t' guaranteed by Proposition 2.4. Again, since u has degree 1, we can assume that $u \notin S$. Let $S^{i_0} \subseteq V(H^{i_0})$ be the i_0-th copy of S. Obviously, S^{i_0} is a cutset in $G_t - e$ and

$$\omega\big((G_t - e) - S^{i_0}\big) = \omega\big((H_t' - e) - S\big) > \frac{|S|}{t} = \frac{|S^{i_0}|}{t},$$

which means that $\tau(G_t - e) < t$.

Therefore, the graph G_t is minimally t-tough.

Now we show that if G_t is minimally t-tough, then G is almost minimally 1-tough.

First, we prove that $\tau(G) \geq 1$. Suppose to the contrary that $\tau(G) < 1$. Obviously, G must be connected (otherwise $\tau(G_t) = 0 \neq t$), so there exists a cutset $S \subseteq V(G)$ in G satisfying

$$\omega(G - S) > |S|.$$

For all $i \in [n]$ let $S_t^i \subseteq V(H^i)$ be the i-th copy of the tough set S_t defined in Notation 4.22. (Note that $v \in S_t$ and $u \notin S_t$.) Let

$$J = J(S) = \big\{ i \in [n] \;\big|\; v_i \in S \big\}$$

and consider the vertex set

$$S' = S \cup \left(\bigcup_{i \in J} S_t^i \right) = \bigcup_{i \in J} S_t^i.$$

Then S' is a cutset in G_t with

$$|S'| = \sum_{i \in J} |S_t^i| = |S| \cdot |S_t|$$

and
$$\omega(G_t - S') > |S| + |S|\left(\frac{|S_t|}{t} - 1\right) = \frac{|S| \cdot |S_t|}{t} = \frac{|S'|}{t},$$
which means that $\tau(G_t) < t$ and that is a contradiction. So $\tau(G) \geq 1$.

Now we prove that $\tau(G - e) < 1$ for all $e \in E(G)$. Let $e \in E(G)$ be an arbitrary edge. If e is a bridge in G, then $\tau(G - e) = 0 < 1$. Let us assume that e is not a bridge in G. Then e is not a bridge in G_t either. Let $S = S(e) \neq \emptyset$ be a vertex set guaranteed by Proposition 2.4. Consider the vertex set $S_0 = S \cap V(G)$. Since e is a bridge in $G - S_0$ as well, S_0 is a cutset in $G - e$. Let
$$C = C(S) = \big\{i \in [n] \;\big|\; v_i \in S_0\big\},$$
$$c_i = \big|V(H^i) \cap S\big| - 1$$
for all $i \in C$ and
$$d_i = \big|V(H^i) \cap S\big|$$
for all $i \in [n] \setminus C$. Let
$$D = D(S) = \big\{i \in [n] \setminus C \;\big|\; d_i > 0\big\}.$$
Then
$$\omega\big((G - e) - S_0\big) > |S_0| = |C|$$
must hold, otherwise, similarly as before,
$$\begin{aligned}\omega\big((G' - e) - S\big) &\leq |C| + \sum_{i \in C}\left(\frac{c_i + 1}{t} - 1\right) + \sum_{i \in D}\left(\frac{d_i}{t} - 1\right)\\ &= \frac{|C| + \sum_{i \in C} c_i + \sum_{i \in D} d_i}{t} - |D| \leq \frac{|S|}{t},\end{aligned}$$
which is a contradiction. So $\tau(G - e) < 1$.

Therefore, G is almost minimally 1-tough. $\square$

Chapter 5

Strengthening some results on toughness of bipartite graphs

Here we prove some complexity results regarding bipartite graphs and we also prove that every minimally 1-tough, bipartite graph on n vertices has a vertex of degree at most $(n+6)/4$.

5.1 Auxiliary results

Claim 5.1. For any positive rational number t the problem EXACT-t-TOUGH belongs to DP.

Proof of Claim 5.1. For any positive rational number t,

$$\begin{aligned}\text{EXACT-}t\text{-TOUGH} &= \{G \text{ graph} \mid \tau(G) = t\} \\ &= \{G \text{ graph} \mid \tau(G) \geq t\} \cap \{G \text{ graph} \mid \tau(G) \leq t\}.\end{aligned}$$

Let

$$L_1 = \{G \text{ graph} \mid \tau(G) \leq t\}$$

and

$$L_2 = \{G \text{ graph} \mid \tau(G) \geq t\}.$$

As we saw in the proof of Proposition 5.1, the language L_1 belongs to NP and L_2 belongs to coNP.

Hence, we can conclude that EXACT-t-TOUGH $= L_1 \cap L_2 \in$ DP. $\square$

For any positive rational number t, let EXACT-t-TOUGH-BIPARTITE denote the problem of determining whether a given bipartite graph has toughness t. Since the toughness of a bipartite graph is at most 1 (except for the graphs K_1 and K_2), we can conclude the following.

Corollary 5.2. For any positive rational number $t \leq 1$, the problem EXACT-t-TOUGH-BIPARTITE belongs to DP. Moreover, the problem EXACT-1-TOUGH-BIPARTITE belongs to coNP.

Theorem 5.3 (Katona, Varga, [4])**.** For any positive rational number t, the problem EXACT-t-TOUGH is DP-complete.

Proof. In Claim 5.1 we already proved that EXACT-t-TOUGH $\in$ DP. To prove EXACT-t-TOUGH is DP-hard, we reduce EXACTINDEPENDENCENUMBER to it.

Let G be an arbitrary connected graph on the vertices $v_1, \dots, v_n$ and let a, b be positive integers such that $t = a/b$. Let k be a positive integer and let G_k be the following graph. For all $i \in [n]$ let

$$V_i = \{v_{i,1}, v_{i,2}, \dots, v_{i,a}\},$$

and let

$$V = \bigcup_{i=1}^{n} V_i, \qquad U = \bigcup_{i=1}^{n} \bigcup_{j=1}^{b} u_{i,j}, \qquad U' = \{u'_1, \dots, u'_{(b-1)k}\},$$
$$W = \{w_1, \dots, w_{ak}\}, \qquad V(G_k) = V \cup U \cup U' \cup W.$$

For all $i \in [n]$ place a clique on V_i. For all $i_1, i_2 \in [n]$ if $v_{i_1} v_{i_2} \in E(G)$, then place a complete bipartite graph on $(V_{i_1}; V_{i_2})$. For all $i \in [n]$ and $j \in [b]$ connect $u_{i,j}$ to every vertex of V_i. Place a clique on W and connect every vertex of W to every vertex of $V \cup U \cup U'$. See Figure 5.1.

Obviously, G_k can be constructed from G in polynomial time. Now we show that $\alpha(G) = k$ if and only if $\tau(G_k) = t = a/b$, i.e.

– if $\alpha(G) > k$, then
$$\frac{|S|}{\omega(G_k - S)} > t$$
for any cutset S of G_k;

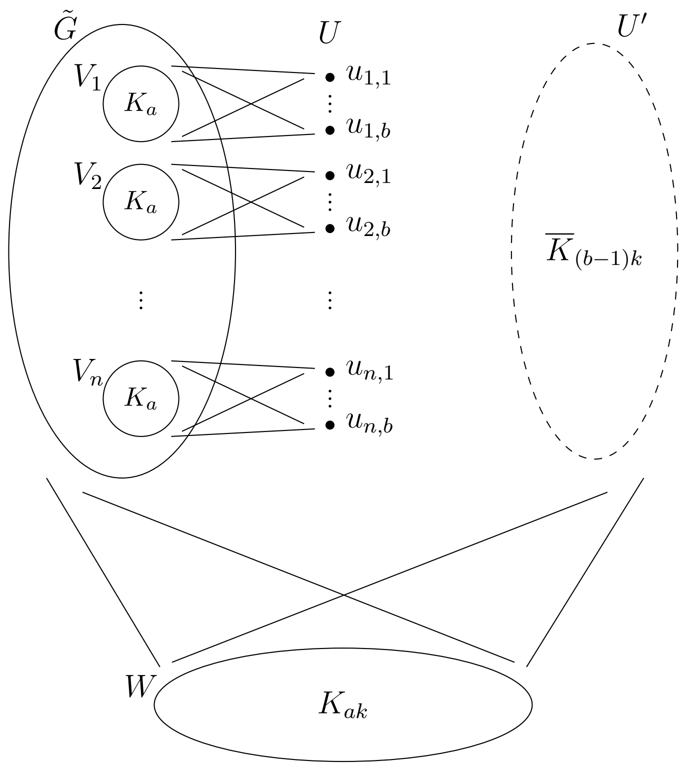

Figure 5.1: The graph G_k.

– if $\alpha(G) < k$, then there exists a cutset S_0 of G_k such that

$$\frac{|S_0|}{\omega(G_k - S_0)} < t;$$

– if $\alpha(G) = k$, then

$$\frac{|S|}{\omega(G_k - S)} > t$$

for any cutset S of G_k and there exists a cutset S_0 of G_k such that

$$\frac{|S_0|}{\omega(G_k - S_0)} < t.$$

Let $S \subseteq V(G_k)$ be an arbitrary cutset of G_k. Since S is a cutset, it must contain W. Let

$$I = \{i \in [n] \mid V_i \subseteq S\}.$$

After the removal of W, the removal of any vertex of $U \cup U'$ or the removal of only a proper subset of V_i for any $i \in [n]$ does not disconnect anything in the graph. So consider the cutset

$$S' = S \setminus \left[(U \cup U') \cup \left(\bigcup_{i \notin I} V_i \right) \right].$$

In $G_k - S'$ there are two types of components: isolated vertices from $U \cup U'$ and components containing at least one vertex from V. There are at most $\alpha(G)$ components of the second type since picking a vertex from each such component forms an independent set of $G[V]$. On the other hand, there are exactly $b|I| + |U'| = b|I| + (b-1)k$ components of the first type. So

$$|S| \geq |S'| = \sum_{i \in I} |V_i| + |W| = a|I| + ak = a\big(|I| + k\big)$$

and

$$\omega(G_k - S) = \omega(G_k - S') \leq \alpha(G) + b|I| + (b-1)k = b\big(|I| + k\big) + \big(\alpha(G) - k\big).$$

Therefore,

$$\frac{|S|}{\omega(G_k - S)} \geq \frac{|S'|}{\omega(G_k - S')} \geq \frac{a\big(|I| + k\big)}{b\big(|I| + k\big) + \big(\alpha(G) - k\big)}.$$

Let $\{v_j \in V(G) \mid j \in J\}$ be an independent set of size $\alpha(G)$ in the graph G for some $J \subseteq [n]$, and consider another cutset

$$S_0 = \left(\bigcup_{i \notin J} V_i \right) \cup W$$

in G_k. Then

$$|S_0| = a\big(n - \alpha(G)\big) + ak = a\big(n - \alpha(G) + k\big)$$

and (similarly as before)

$$\omega(G_k - S_0) = \alpha(G) + b\big(n - \alpha(G)\big) + (b-1)k = b\big(n - \alpha(G) + k\big) + \big(\alpha(G) - k\big),$$

so

$$\frac{|S_0|}{\omega(G_k - S_0)} = \frac{a\big(n - \alpha(G) + k\big)}{b\big(n - \alpha(G) + k\big) + \big(\alpha(G) - k\big)}.$$

Case 1: $\alpha(G) < k$.
Then

$$\frac{|S|}{\omega(G_k - S)} \geq \frac{a\big(|I| + k\big)}{b\big(|I| + k\big) + \big(\alpha(G) - k\big)} > \frac{a\big(|I| + k\big)}{b\big(|I| + k\big)} = \frac{a}{b} = t$$

holds for every cutset S of G_k, which implies that $\tau(G_k) > t$.

Case 2: $\alpha(G) = k$.
Then

$$\frac{|S|}{\omega(G_k - S)} \geq \frac{a\big(|I| + k\big)}{b\big(|I| + k\big) + \big(\alpha(G) - k\big)} = \frac{a\big(|I| + k\big)}{b\big(|I| + k\big)} = \frac{a}{b} = t$$

holds for every cutset S of G_k, which implies that $\tau(G_k) \geq t$.

On the other hand,

$$\tau(G_k) \leq \frac{|S_0|}{\omega(G_k - S_0)} = \frac{a\big(n - \alpha(G) + k\big)}{b\big(n - \alpha(G) + k\big) + \big(\alpha(G) - k\big)} = \frac{an}{bn} = \frac{a}{b} = t.$$

Hence, $\tau(G_k) = t$.

Case 3: $\alpha(G) > k$.
Then

$$\tau(G_k) \leq \frac{|S_0|}{\omega(G_k - S_0)} = \frac{a\big(n - \alpha(G) + k\big)}{b\big(n - \alpha(G) + k\big) + \big(\alpha(G) - k\big)} < \frac{a\big(n - \alpha(G) + k\big)}{b\big(n - \alpha(G) + k\big)} = \frac{a}{b} = t.$$

This means that $\alpha(G) = k$ if and only if $\tau(G_k) = t = a/b$. $\square$

The construction we used here is a slight modification of the one that Bauer et al. used in [11] for proving that for any rational number $t \geq 1$ recognizing t-tough graphs is coNP-complete; as mentioned earlier, in their proof a variant of INDEPENDENCENUMBER is reduced to the complement of t-TOUGH.

Since in our proof $\alpha(G) > k$ if and only if $\tau(G_k) < t$, we can reduce INDEPENDENCENUMBER to the complement of t-TOUGH, therefore providing another proof of Theorem 2.8.

5.2 The complexity of recognizing t-tough bipartite graphs

Here we extend Theorem 2.10 to any positive rational number $t \leq 1$. (As mentioned in the previous section, the toughness of of a bipartite graph is at most 1 except for the graphs K_1 and K_2.)

Theorem 5.4 (Katona, Varga, [4])**.** For any positive rational number $t \leq 1$, the problem t-Tough remains coNP-complete for bipartite graphs.

First, we need some preparation.

Proposition 5.5. Let $G \ncong K_1, K_2$ be a 1/2-tough graph. Then there exists a spanning subgraph H of G for which $\tau(H) = 1/2$.

Proof. Let H be a spanning subgraph of G so that $\tau(H) \geq 1/2$ and there exists an edge $e \in E(H)$ for which $\tau(H - e) < 1/2$. (Note that since $\tau(G) \geq 1/2$, such a spanning subgraph H can be obtained by repeatedly deleting some edges of G.)

Now we show that $\tau(H) \leq 1/2$, which implies that $\tau(H) = 1/2$. Let $e \in E(G)$ be an edge for which $\tau(H - e) < 1/2$.

Case 1: e is a bridge in H.
Since G is 1/2-tough, it is connected. Since $G \ncong K_1, K_2$ and G is connected, the graphs G and H have at least three vertices. Hence, at least one of the endpoints of e is a cut-vertex in H, so $\tau(H) \leq 1/2$.

Case 2: e is not a bridge in H.
Then there exists a cutset S in $H - e$ for which

$$\omega\big((H - e) - S\big) > 2|S|.$$

Case 2.1: (e is not a bridge in H) and S is a cutset in H.
Then

$$\omega(H - S) \leq 2|S|,$$

which is only possible if

$$\omega(H - S) = 2|S| \qquad \text{and} \qquad \omega\big((H - e) - S\big) = 2|S| + 1.$$

Therefore, $\tau(H) \leq 1/2$.

Case 2.2: (e is not a bridge in H) and S is not a cutset in H.
This is only possible if

$$\omega\big((H-e)-S\big) = 2.$$

Hence

$$2 = \omega\big((H-e)-S\big) > 2|S|,$$

i.e. $|S| < 1$, which means that $S = \emptyset$, so e is a bridge H, which is a contradiction. □

Let G be an arbitrary connected graph on the vertices $v_1, \dots, v_n$ and let $B(G)$ be the following bipartite graph. Let

$$V\big(B(G)\big) = \big\{v_{i,1}, v_{i,2} \mid i \in [n]\big\}$$

and for all $i, j \in [n]$ if $v_i v_j \in E(G)$, then connect $v_{i,1}$ to $v_{j,2}$ and $v_{i,2}$ to $v_{j,1}$. Also for all $i \in [n]$ connect $v_{i,1}$ to $v_{i,2}$. See Figure 5.2.

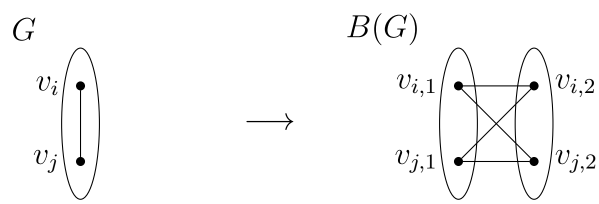

Figure 5.2: The construction of the bipartite graph $B(G)$.

Now we show how the toughness of $B(G)$ can be computed from the toughness of G.

Claim 5.6. Let G be an arbitrary connected graph. Then $\tau\big(B(G)\big) = \min\big(2\tau(G), 1\big)$.

Proof. Let G be an arbitrary graph on the vertices $v_1, \dots, v_n$ with $\tau(G) = t$.

Case 1: $t \leq 1/2$.

Let $G' = B(G)$ and let $S_0 \subseteq V(G)$ be an arbitrary tough set in G. (Note that since $\tau(G) \le 1/2$, the graph G is noncomplete, therefore it has a tough set.) Consider the vertex set

$$S_0' = \{v_{i,1}, v_{i,2} \mid v_i \in S_0\}.$$

Clearly, S_0' is a cutset in G' and

$$\omega(G' - S_0') = \omega(G - S_0) = \frac{|S_0|}{t} = \frac{|S_0'|}{2t},$$

so $\tau(G') \le 2t$.

Now we prove that $\tau(G') \ge 2t$, i.e.

$$\omega(G' - S') \le \frac{|S'|}{2t}$$

holds for any cutset S' of G'. Therefore, let S' be an arbitrary cutset in G' and let

$$S_1' = \{v_{i,1} \in S' \mid v_{i,2} \notin S'\} \cup \{v_{i,2} \in S' \mid v_{i,1} \notin S'\}$$

and

$$S_2' = S' \setminus S_1'.$$

Consider the components of $G' - S'$ which contain either both or none of the vertices $v_{i,1}, v_{i,2}$ for any $i \in [n]$. These components of $G' - S'$ are also components of $G' - S_2'$, so (similarly as before) the number of these components is at most $|S_2'|/2t$. The number of the remaining components – so in which there is at least one vertex without its pair – can be at most $|S_1'|$, because the pair of the vertex mentioned before must be in S_1'. Since $t \le 1/2$,

$$\omega(G' - S') \le \frac{|S_2'|}{2t} + |S_1'| \le \frac{|S_2'|}{2t} + \frac{|S_1'|}{2t} = \frac{|S'|}{2t},$$

which implies that $\tau(G') \ge 2t$.

Hence,

$$\tau(G') = 2t = 2\tau(G) = \min\big(2\tau(G), 1\big).$$

Case 2: $t > 1/2$.

By Proposition 5.5, there exists a spanning subgraph H with $\tau(H) = 1/2$. Then $B(H)$ is a spanning subgraph of $B(G)$, so

$$\tau\big(B(G)\big) \ge \tau\big(B(H)\big),$$

and as we saw in Case 1,

$$\tau\big(B(H)\big) = 2\tau(H) = 1.$$

Since $B(G)$ is a bipartite graph, $\tau\big(B(H)\big) \leq 1$. Hence,

$$\tau\big(B(G)\big) = 1 = \min\big(2\tau(G), 1\big).$$

□

Theorem 5.7 (Katona, Varga, [4])**.** For any positive rational number $t < 1$, the problem EXACT-t-TOUGH remains DP-complete for bipartite graphs.

Proof. In Corollary 5.2 we already proved that if $t < 1$, then EXACT-t-TOUGH-BIPARTITE $\in$ DP.

Now we reduce the DP-complete problem EXACT-$t/2$-TOUGH to EXACT-t-TOUGH-BIPARTITE if $t < 1$, and the coNP-complete problem $1/2$-TOUGH to (EXACT-)1-TOUGH-BIPARTITE.

Let $t < 1$ be a positive rational number and let G be an arbitrary connected graph. By Claim 5.6,

- $\tau\big(B(G)\big) = t$ if and only if $\tau(G) = t/2$, and
- $\tau\big(B(G)\big) = 1$ if and only if $\tau(G) \geq 1/2$,

thus the statement of the theorem follows. □

Since in Corollary 5.2 we already proved that (EXACT-)1-TOUGH-BIPARTITE $\in$ coNP and Claim 5.6 also holds for $t = 1$, the above proof can be used to give an alternative proof of Theorem 2.10.

It also follows from Theorem 5.7 that recognizing t-tough bipartite graphs is coNP-complete for any positive rational number $t \leq 1$.

Proof of Theorem 5.4. Since in the above proof $\tau\big(B(G)\big) \geq t$ if and only if $\tau(G) \geq t/2$ for any positive rational number $t \leq 1$, we can reduce $t/2$-TOUGH to t-TOUGH-BIPARTITE, so the statement of the theorem follows. □

As mentioned, the case $t = 1$ (i.e., Theorem 2.10) was already proved by Kratsch et al. in [21]. In their proof the vertices $v_{i,1}$ and $v_{i,2}$ are not connected by an edge, but by a path with two inner vertices. With that construction the original graph is 1-tough if and only if the obtained bipartite graph is exactly 1-tough. However, due to the inner vertices of the paths mentioned before, the constructed bipartite graph has a lot of vertices of degree 2, so these graphs are neither regular (except for cycles) nor 3-connected.

Motivated by two open problems regarding the complexity of recognizing 1-tough 3-connected bipartite graphs and 1-tough 3-regular bipartite graphs [9], we also prove the following.

Theorem 5.8 (Katona, Varga, [4])**.** For any fixed integer $k \geq 2$ and positive rational number $t \leq 1$, the problem t-TOUGH remains coNP-complete for k-connected bipartite graphs.

Theorem 5.9 (Katona, Varga, [4])**.** For any fixed integer $r \geq 6$, the problem 1-TOUGH remains coNP-complete for r-regular bipartite graphs.

5.3 The complexity of recognizing t-tough k-connected bipartite graphs where $k \geq 2$ is an integer and $t \leq 1/2$

To prove Theorem 5.8, we only need one more proposition.

Proposition 5.10. Let G be an arbitrary graph. Then $\kappa\big(B(G)\big) \geq \kappa(G)$.

Proof. Let S be an arbitrary cutset in $B(G)$. We need to show that $|S| \geq \kappa(G)$.

Let

$$W = \big\{v_{i,1}, v_{i,2} \;\big|\; \{v_{i,1}, v_{i,2}\} \cap S = \emptyset\big\}.$$

Case 1: the vertices of W belong to at least two components of $B(G)-S$. Then

$$S' = \{v_j \in V(G) \mid v_{j,1}, v_{j,2} \notin W\}$$

is a cutset in G: its removal from G disconnects the corresponding vertices of W that belong to different components of $B(G) - S$. Obviously,

$$|S| \geq |S'| \geq \kappa(G).$$

Case 2: all vertices of W belong to one component of $B(G) - S$.

Since S is a cutset in $B(G)$, there exists a component L for which $L \cap W = \emptyset$. We can assume that $v_{i,1} \in L$ for some $i \in [n]$. Then $v_{i,2} \in S$ since $L \cap W = \emptyset$.

Also, for every $j \in [n]$, if $v_i v_j \in E(G)$, then either $v_{j,2} \in S$ or $v_{j,2} \in L$, and in the latter case $v_{j,1} \in S$ holds since $L \cap W = \emptyset$. Therefore,

$$|S| \geq d(v_{i,1}) = d(v_i) + 1 \geq \delta(G) + 1 > \kappa(G).$$

Hence, $\kappa\big(B(G)\big) \geq \kappa(G)$. □

Proof of Theorem 5.8. Let $k \geq 2$ be an integer and $t \leq 1$ positive rational number. Applying the proof of Theorem 5.4 for k-connected bipartite graphs, the statement of theorem follows from Proposition 5.10. □

5.4 The complexity of recognizing 1-tough, at least 6-regular, bipartite graphs

For any positive rational number t and positive integer r let t-TOUGH-r-REGULAR denote the problem of determining whether a given r-regular graph is t-tough, and let t-TOUGH-r-REGULAR-BIPARTITE denote the same problem for bipartite graphs.

For any odd number $r \geq 5$ let H_r be the complement of the graph whose vertex set is

$$V = \{w, u_1, \ldots, u_{r+1}\}$$

and whose edge set is

$$E = \left(\bigcup_{i=1}^{\frac{r-1}{2}} \{u_i, u_{r-i+2}\} \right) \cup \{w, u_{(r+1)/2}\} \cup \{w, u_{(r+3)/2}\}.$$

For any even number $r \geq 6$ let H_r be a bipartite graph with color classes

$$A = \{w_a, a_1, \ldots, a_{r-1}\} \qquad \text{and} \qquad B = \{w_b, b_1, \ldots, b_{r-1}\},$$

which can be obtained from the complete bipartite graph by removing the edge $\{w_a, w_b\}$. (See the graphs $\overline{H}_5$, H_5 and H_6 in Figure 5.3.)

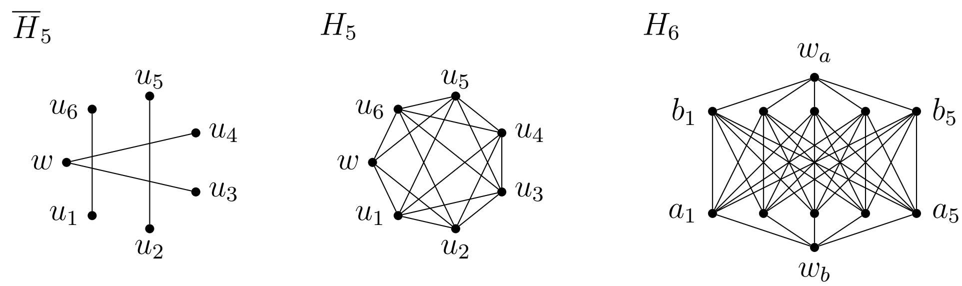

Figure 5.3: The graphs $\overline{H}_5$, H_5 and H_6.

Claim 5.11. Let $r \geq 5$ be an integer. Then $\tau(H_r) \geq 1$.

Proof. There is a Hamiltonian cycle in H_r, namely

$$wu_1u_2 \dots u_{r+1}w$$

if r is odd, and

$$w_a b_1 a_1 w_b a_2 b_2 a_3 b_3 \dots a_{r-1} b_{r-1} w_a$$

if r is even, so $\tau(H_r) \geq 1$. □

Lemma 5.12 (Katona, Varga, [4]). For any fixed odd number $r \geq 5$ the problem 1/2-Tough is coNP-complete for r-regular graphs.

Proof. Obviously, 1/2-Tough-r-Regular $\in$ coNP. To prove that it is coNP-hard, we reduce 1-Tough-$(r-1)$-Regular to it.

Let G be an arbitrary connected $(r-1)$-regular graph on the vertices $v_1, \dots, v_n$ and let G' be defined as follows. For all $i \in [n]$ let

$$V_i = \{w^i, u_1^i, \dots, u_{r+1}^i\}$$

and place the graph H_r on the vertices of V_i and also connect v_i to w^i, see Figure 5.4. It is easy to see that G' is r-regular and can be constructed from G in polynomial time. Now we prove that G is 1-tough if and only if G' is 1/2-tough.

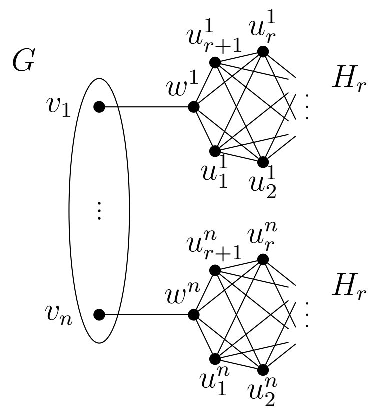

Figure 5.4: The graph G' constructed in the proof of Lemma 5.12.

If G is not 1-tough, then there exists a cutset $S \subseteq V(G)$ satisfying $\omega(G - S) > |S|$. Then S is also a cutset in G' and

$$\omega(G' - S) = \omega(G - S) + |S| > 2|S|,$$

so $\tau(G') < 1/2$.

Now assume that G is 1-tough. Let $S \subseteq V(G')$ be an arbitrary cutset in G', and let $S_0 = V(G) \cap S$ and $S_i = V_i \cap S$ for all $i \in [n]$. Using these notations it is clear that

$$S = S_0 \cup \left(\bigcup_{i=1}^{n} S_i \right)$$

and

$$\omega(G' - S) \leq \omega\big(G - S_0\big) + |S_0| + \sum_{i=1}^{n} \omega(H_r^i - S_i),$$

where H_r^i denotes the i-th copy of H_r, i.e. the graph on the vertex set V_i for all $i \in [n]$. Since G is 1-tough and by Claim 5.11, so is H_r, it follows from Proposition 2.2 that

$$\omega(G - S_0) \leq |S_0|$$

and

$$\omega(H_r^i - S_i) \leq |S_i|.$$

Therefore,

$$\omega(G' - S) \leq |S_0| + |S_0| + \sum_{i=1}^{n} |S_i| \leq 2|S|,$$

so $\tau(G') \geq 1/2$. □

Lemma 5.13 (Katona, Varga, [4])**.** For any fixed even number $r \geq 6$ the problem 1/2-TOUGH is coNP-complete for r-regular graphs.

Proof. Obviously, 1/2-TOUGH-r-REGULAR $\in$ coNP. To prove that it is coNP-hard we reduce 1-TOUGH-$(r-2)$-REGULAR to it.

Let G be an arbitrary connected $(r-2)$-regular graph on the vertices $v_1, \ldots, v_n$ and let G' be defined as follows. For all $i \in [n]$ let

$$A_i = \{w_a^i, a_1^i, \ldots, a_{r-1}^i\}, \qquad B_i = \{w_b^i, b_1^i, \ldots, b_{r-1}^i\}$$

and place the graph H_r on the color classes A_i and B_i and also connect v_i to w_a^i and w_b^i, see Figure 5.5. It is easy to see that G' is r-regular and can be constructed from G in polynomial time.

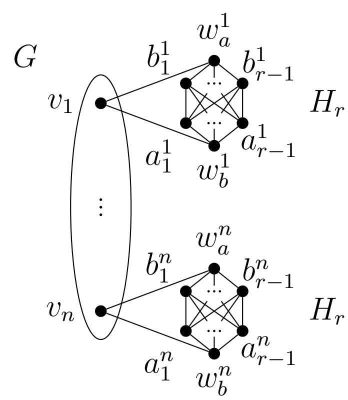

Figure 5.5: The graph G' constructed in the proof of Lemma 5.13.

Similarly as in the proof of Lemma 5.12, it can be shown that G is 1-tough if and only if G' is 1/2-tough. □

Corollary 5.14 (Katona, Varga, [4])**.** For any fixed integer $r \geq 5$ the problem 1/2-TOUGH is coNP-complete for r-regular graphs.

Using this result, we can prove Theorem 5.9.

Proof of Theorem 5.9. Obviously, 1-TOUGH-r-REGULAR-BIPARTITE $\in$ coNP. To prove that it is coNP-hard we reduce $1/2$-TOUGH-$(r-1)$-REGULAR to it.

Let G be an arbitrary connected $(r-1)$-regular graph and let $B(G)$ denote the bipartite graph defined at the beginning of Section 5.2. Then $B(G)$ is r-regular and by Claim 5.6, the graph G is $1/2$-tough if and only if $B(G)$ is 1-tough. $\square$

For any $r \in \{3,4,5\}$ the problem of determining the complexity of 1-TOUGH-r-REGULAR-BIPARTITE remains open. We note that the reason why our construction does not work in these cases is that we can decide in polynomial time whether an at most 4 regular graph is $1/2$-tough.

Theorem 5.15 (Katona, Varga, [4])**.** For any positive rational number $t < 2/3$ there is a polynomial time algorithm to recognize t-tough 3-regular graphs.

To prove this theorem, we need the following lemma.

Lemma 5.16 (Katona, Varga, [4])**.** For any connected 3-regular graph G, the following are equivalent.

(1) There is a cut-vertex in G.

(2) $\tau(G) \leq 1/2$.

(3) $\tau(G) < 2/3$.

Proof.
(1) $\Longrightarrow$ (2) : Trivial.

(2) $\Longrightarrow$ (3) : Trivial.

(3) $\Longrightarrow$ (1) : If $\tau(G) < 2/3$, then there exists a cutset $S \subseteq V(G)$ satisfying

$$\omega(G-S) > \frac{3}{2}|S|.$$

Hence there must exist a component of $G-S$ that has exactly one neighbor in S: since G is connected, every component has at least one neighbor in S, and if every component of $G-S$ had at least two neighbors in S, then the number of edges going into S would be at least $2\omega(G-S) > 3|S|$, which would contradict the 3-regularity of G. Obviously, this neighbor in S is a cut-vertex in G. $\square$

Proof of Theorem 5.15. Let G be an arbitrary connected 3-regular graph. First check whether G contains a cut-vertex. By Lemma 5.16, if it does not, then $\tau(G) \geq 2/3$, but if it does, then $\tau(G) \leq 1/2$. We prove that in the latter case either $\tau(G) = 1/3$ or $\tau(G) = 1/2$, and we can also decide in polynomial time which one holds.

Since G is 3-regular, $\omega(G-S) \leq 3|S|$ holds for any cutset S of G, so $\tau(G) \geq 1/3$. Now we show that if $\tau(G) < 1/2$, then $\tau(G) \leq 1/3$.

So assume that $\tau(G) < 1/2$ and let S be a tough set of G and let $k = \omega(G-S)$. Then $k > 2|S|$. Contract the components of $G-S$ into single vertices $u_1, \dots, u_k$ while keeping the multiple edges and let H denote the obtained multigraph. Since G is connected, $d(u_i) \geq 1$ holds for any $i \in [k]$, so

$$k = \big|\{i \in [k] : d(u_i) = 1\}\big| + \big|\{i \in [k] : d(u_i) \geq 2\}\big|.$$

Since G is 3-regular,

$$\begin{aligned} 3|S| &\geq \sum_{i=1}^{k} d(u_i) \geq \big|\{i \in [k] : d(u_i) = 1\}\big| + 2 \cdot \big|\{i \in [k] : d(u_i) \geq 2\}\big| \\ &= k + \big|\{i \in [k] : d(u_i) \geq 2\}\big| > 2|S| + \big|\{i \in [k] : d(u_i) \geq 2\}\big|, \end{aligned}$$

so

$$|S| > \big|\{i \in [k] : d(u_i) \geq 2\}\big|.$$

Therefore,

$$\big|\{i \in [k] : d(u_i) = 1\}\big| = k - \big|\{i \in [k] : d(u_i) \geq 2\}\big| > 2|S| - |S| = |S|,$$

which means that there exists a vertex in S having at least two neighbors in $\{u_1, \dots, u_k\}$ of degree 1. Then the removal of this vertex leaves at least three components (and note that since G is 3-regular, it cannot leave more than three components), so $\tau(G) \leq 1/3$.

From this it also follows that $\tau(G) = 1/3$ if and only if there exists a cut-vertex whose removal leaves three components.

To summarize, it can be decided in polynomial time whether a connected 3-regular graph is 2/3-tough, and if it is not, then its toughness is either

1/3 or 1/2. In both cases the graph contains at least one cut-vertex, and if the removal of any of them leaves (at least) three components, then the toughness of the graph is 1/3, otherwise it is 1/2. □

Theorem 5.17 (Katona, Varga, [4])**.** There is a polynomial time algorithm to recognize 1/2-tough 4-regular graphs.

The proof of this theorem follows directly from the following claim.

Claim 5.18 (Katona, Varga, [4])**.** The toughness of any connected 4-regular graph is at least 1/2.

Proof. Let G be a connected 4-regular graph and let S be an arbitrary cutset in G and L be a component of $G - S$. Since every vertex has degree 4 in G, the number of edges between S and L is even (more precisely, it is equal to the sum of the degrees in G of the vertices of L minus two times the number of edges induced by L). Since G is connected, the number of these edges is at least two. On the other hand, since G is 4-regular, there are at most $4|S|$ edges between S and L. Therefore $\omega(G - S) \leq 2|S|$, which means that $\tau(G) \geq 1/2$. □

5.5 Upper bound on the minimum degree of minimally 1-tough, bipartite graphs

Finally, we strengthen the upper bound we gave in Theorem 3.3 for minimally 1-tough, bipartite graphs.

Theorem 5.19 (Katona, Varga)**.** Every minimally 1-tough, bipartite graph on n vertices has a vertex of degree at most $(n + 6)/4$.

Proof of Theorem 5.19. Suppose to the contrary that $\delta(G) > (n+6)/4$. Obviously, a 1-tough bipartite graph must be balanced and therefore the number of its vertices must be even, hence $\delta(G) \geq n/4 + 2$. Consider an arbitrary edge $e = uv$. By Propostition 2.4, there exists a vertex set $S = S(e)$, whose removal from $G-e$ leaves $|S|+1$ connected components, and u and v belong to different components. Let $k = |S|$ and let L_u and L_v denote the components of u and v, respectively. Obviously, the components of $(G - e) - S$ require

$$\omega\big((G - e) - S\big) = k + 1$$

independent vertices: two of them can be a and b, but the rest of them cannot be adjacent either to a or to b, and the rest of them cannot be either in $L_u \cup L_v$ or in S.

Since there are no triangles in the graph, the open neighborhood of u and v contains at least

$$2 \cdot \left(\frac{n}{4} + 2\right) - 2 = \frac{n}{2} + 2$$

vertices and at most k of them belongs to S, so at least

$$\left(\frac{n}{2} + 2\right) - k$$

vertices belong to $L_u \cup L_v$.

Since G is bipartite, the components of $G - S$ are also bipartite and since G is 1-tough, the sizes of the two color classes of these components can differ in at most 1. Therefore,

$$|L_u| + |L_v| \geq 2 \cdot \left(\frac{n}{2} + 2 - k\right) - 2 = n - 2k + 2.$$

Hence, for the remaining $k - 1$ components there are only

$$n - ((n - 2k + 2) + k) = k - 2$$

vertices, which is a contradiction. $\square$

Summary

The main focus of the thesis is on minimally tough graphs.

Chapter 3 is motivated by a conjecture, saying that every minimally 1-tough graph on n vertices has a vertex of degree 2. In this chapter, we give an upper bound on the minimum degree of minimally 1-tough graphs, namely $n/3 + 1$.

Chapter 4 investigates the complexity of recognizing minimally t-tough graphs. There we prove that this problem is DP-hard for all positive rational number t.

In Chapter 5, we study bipartite graphs. First, we show that recognizing t-tough bipartite graphs is coNP-hard for all positive rational number $t \leq 1$ (the case $t = 1$ was already known). Motivated by two open problems regarding the complexity of recognizing 1-tough 3-connected bipartite graphs and 1-tough 3-regular bipartite graphs, we also prove that recognizing t-tough k-connected bipartite graphs and 1-tough r-regular bipartite graphs is also coNP-complete for any integers $k \geq 2$ and $r \geq 6$ and for any positive rational number $t \leq 1$.

Appendix A

Appendix

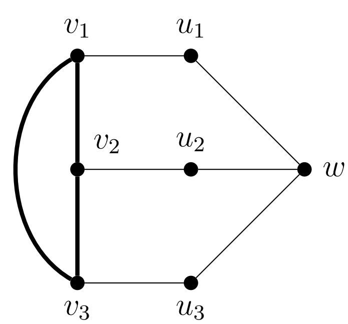

Figure A.1: The minimally 1-tough graph G' constructed in the beginning of Section 4.2, when $G \simeq K_3$. The edges of K_3 are drawn with thick lines.

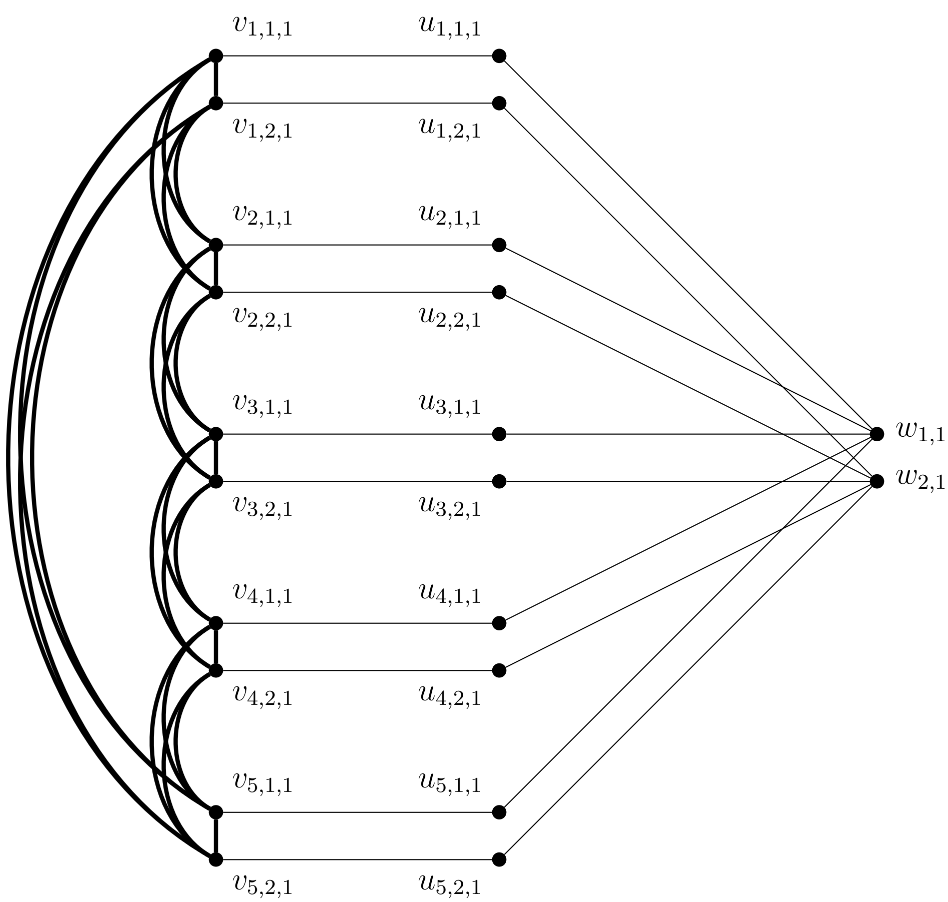

Figure A.2: The graph $G'_{1,2}$ constructed in Subsection 4.2.1, when $G \simeq C_5$. Since the graph C_5 is connected and α-critical with $\alpha(C_5) = 2$, the choice $k = 2$ results in a minimally 1-tough graph. The edges of the "blown-up" C_5 are drawn with thick lines.

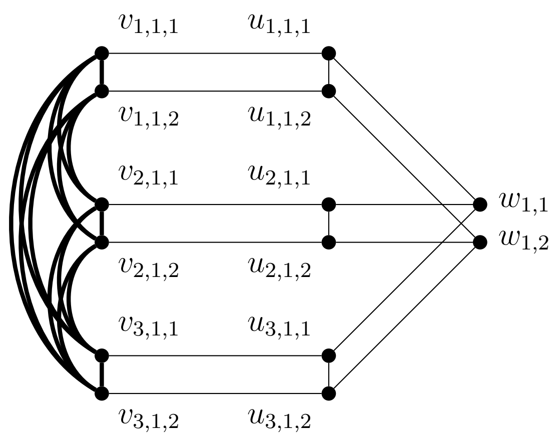

Figure A.3: The graph $G'_{2,1}$ constructed in Subsection 4.2.1, when $G \simeq K_3$. Since the graph K_3 is 2-connected and α-critical with $\alpha(K_3) = 1$, the choice $k = 1$ results in a minimally 2-tough graph. The edges of the "blown-up" K_3 are drawn with thick lines.

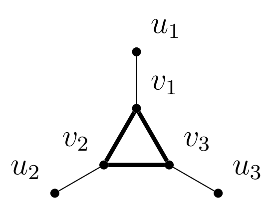

Figure A.4: The graph $G_{1/2}$ constructed in Subsection 4.2.2, when $G \simeq K_3$. Since the graph K_3 is almost minimally 1-tough, this graph is minimally 1/2-tough. The edges of K_3 are drawn with thick lines.

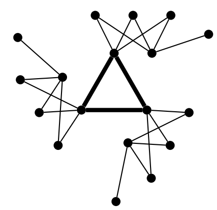

Figure A.5: The graph $G_{2/5}$ constructed in Subsection 4.5.3, when $G \simeq K_3$. Since the graph K_3 is almost minimally 1-tough, this graph is minimally 2/5-tough. The edges of K_3 are drawn with thick lines.

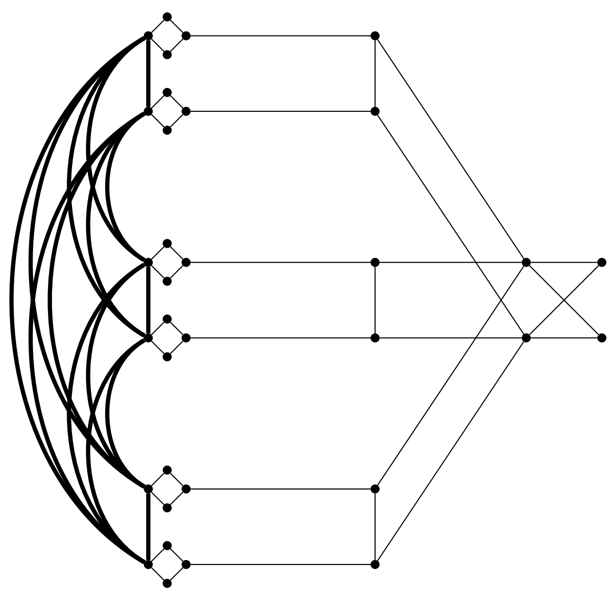

Figure A.6: The graph $G_{2/3,1}$ constructed in Subsection 4.3.3, when $G \simeq K_3$. Since the graph K_3 is 2-connected and α-critical with $\alpha(K_3) = 1$, the choice $k = 1$ results in a minimally 2/3-tough graph. The edges of the "blown-up" K_3 are drawn with thick lines.

List of Publications

[1] M. Kano, G. Y. Katona, and K. Varga, *Decomposition of a graph into two disjoint odd subgraphs*, Graphs and Combinatorics **34 (6)** (2018), 1581–1588.

[2] G. Y. Katona, I. Kovács, and K. Varga, *The complexity of recognizing minimally t-tough graphs*, Discrete Applied Mathematics **294** (2021), 55–84.

[3] G. Y. Katona, D. Soltész, and K. Varga, *Properties of minimally t-tough graphs*, Discrete Mathematics **341** (2018), 221–231.

[4] G. Y. Katona and K. Varga, *Strengthening some complexity results on toughness of graphs*, Discussiones Mathematicae, posted on 2020, DOI 10.7151/dmgt.2372.

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